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Higher-dimensional Rotating Charged Black Holes

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ABSTRACT

Using the blackfold approach, we study new classes of higher-dimensional rotating black holes with electric charges and string dipoles, in theories of gravity coupled to a 2-form or 3-form field strength and to a dilaton with arbitrary coupling. The method allows to describe not only black holes with large angular momenta, but also other regimes that include charged black holes near extremality with slow rotation. We construct explicit examples of electric rotating black holes of dilatonic and non-dilatonic Einstein-Maxwell theory, with horizons of spherical and non-spherical topology. We also find new families of solutions with string dipoles, including a new class of prolate black rings. Whenever there are exact solutions that we can compare to, their properties in the appropriate regime are reproduced precisely by our solutions. The analysis of blackfolds with string charges requires the formulation of the dynamics of anisotropic fluids with conserved string-number currents, which is new, and is carried out in detail for perfect fluids. Finally, our results indicate new instabilities of near-extremal, slowly rotating charged black holes, and motivate conjectures about topological constraints on dipole hair.

Contents

1 Introduction

Charged black holes in dimensions higher than four play an important role in supergravity and in string theory, and large families of them are explicitly known. If they are also rotating, we expect their dynamics to become at least as rich as for neutral black holes [1, 2]. However, the number of known exact solutions with both charge and rotation is considerably smaller than in the absence of either of them. Restricting to theories with a single gauge field, possibly coupled to a dilaton (and without a cosmological constant), the known exact black hole solutions[1] with electric charge reduce, in five dimensions, to the BMPV solution and its non-extremal extension [3] plus the black rings of [4, 5, 6, 7] (with a Chern-Simons term). In $D > 5$ there are the charged black holes that can be obtained by U-duality-type transformations of the Myers-Perry black holes [8], but these only allow a few values of the dilaton coupling, in particular the Kaluza-Klein value [9], but never zero coupling. In fact, a solution whose exact form is conspicuously absent is the natural higher-dimensional extension of the Kerr-Newman solution, namely a rotating charged black hole of the Einstein-Maxwell theory, without dilaton coupling nor Chern-Simons terms.

Given the plethora of novel black hole geometries and topologies uncovered in recent years, many new classes of charged rotating solutions can be anticipated. Some progress has been achieved perturbatively, by adding a small rotation to a charged static black hole [10], or a small charge to a rotating Myers-Perry black hole or black ring [11]. Numerical analysis has also been used to find five-dimensional black holes with one angular momentum, or odd-dimensional black holes with equal-magnitude angular momenta [12]. Note that none of these black holes reach an ultraspinning regime, and their horizons always have spherical topology.

In this paper we follow another approach that also produces approximate solutions, but whose starting point is completely different. Neither the amount of charge nor the rotation of the black holes need be small. Indeed these solutions can be in the ultraspinning regime characteristic of higher-dimensional black holes, but also in other regimes where the mass, charge, and angular momenta enter in different proportions. A regime that is particularly new for the method we use is one where the charge is close to extremality and the rotation is very slow. Both spherical and non-spherical horizon topologies appear naturally in all regimes. And perhaps most remarkably, to leading order in the approximation the solutions are obtained very easily in analytic form in many cases of interest, and the calculation of their physical properties is straightforward and often very simple.

The method also lends itself to the study of black holes with dipoles. In $D \geq 5$ a black hole

[1] We only consider asymptotically flat black holes that are regular on and outside a connected event horizon.

can act as an electric source for a $(q+2)$-form field strength $H_{[q+2]}$, in a theory of the form

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - \frac{1}{2(q+2)!}e^{-2a\phi}H^2_{[q+2]}\right) . \tag{1.1}$$

However, if the spacetime is asymptotically flat, only when $q = 0$ can the black hole possess a conserved monopolar electric charge. When $q \geq 1$ no conserved charge can be associated to its field. Instead, the black hole carries a dipole. The first such solutions were found in [13] in the form of five-dimensional dipole black rings. In fact they remain the only known dipole black holes of (1.1) with $q \geq 1$,[2] but it is to be expected that large classes of dipole black holes exist, in particular in $D \geq 6$.[3] The only approach developed so far that allows to investigate them systematically is the one employed here (other particular instances of the method have been studied in [16], and will also be presented in [17]).

In this paper we only consider the theories (1.1) with $q = 0$ or $q = 1$, so the black holes have 0-brane (*i.e.,* particle) charge, or 1-brane (*i.e.,* string) dipole[4]. A main reason is that this allows us to perform a very complete and explicit study of generic properties of the solutions. The dilaton coupling a remains arbitrary, so in particular our study includes the pure Einstein-Maxwell theory $q = 0$, $a = 0$. Values $q > 1$ will be investigated elsewhere.

The method we employ is the *blackfold approach* of [18, 19]. The black hole is constructed as a black p-brane whose worldvolume wraps a submanifold in a background spacetime. The black brane is thin, in the sense that the horizon size r_0 in directions transverse to the worldvolume is much smaller than any characteristic length scale R along the worldvolume. To leading order in r_0/R, the brane is treated as a 'probe' whose backreaction on the background is neglected. At higher orders, backreaction effects can be systematically included by performing a matched asymptotic expansion [20]. However, the leading order approximation often gives enough non-trivial information about stationary solutions. Using this approach refs. [18, 21] have presented large classes of new neutral black holes.

Charges can be naturally incorporated in this framework. The starting point now are black p-brane solutions of the theories (1.1) with charges of q-branes. We can think of the p-brane as having a charge density of objects that extend along q spatial directions, 'dissolved' in its worldvolume. Therefore, when $q = 0$ the p-brane has pointlike electric charges on its worldvolume, and if it wraps a compact submanifold of the background, we obtain a black hole with electric charge. When $q > 0$, a blackfold with compact worldvolume gives a black hole with an electric q-brane dipole, as discussed above. Since the charge introduces one more scale in the problem, the regime of applicability of the method has to be reassessed. We find that when the 0-brane charge is sufficiently large, the black hole need not be ultraspinning in order to resemble locally a black brane. Due to the cancellation

[2]This refers to dipole black holes with a single, connected horizon, and no charge. Refs. [14] present multi-black holes of this theory with dipoles. The five-dimensional rotating black holes and rings of [5, 6, 7, 8, 9] are solutions of of (1.1) with magnetic charge and electric dipole when $a \neq 0$.

[3]However, there are strong restrictions on the possible existence of *static* black holes with dipoles [15].

[4]Our solutions are dual to magnetic sources of fields $H_{[D-q-2]}$, but other than this we do not consider magnetic charges, nor dyonic solutions.

of forces in the charged extremal limit, one can construct brane-like distributions of 'charged dust'. These extremal configurations are singular, but they can be thermally excited above extremality to exhibit regular horizons, and the (small) tensions that appear are then balanced with suitable slow rotations.

The equations that describe the dynamics of blackfolds decompose into two (coupled) sets: intrinsic equations, which take the form of fluid-dynamics on the worldvolume, and extrinsic equations, which characterize the embedding of the worldvolume in the background. The inclusion of charges has the effect that the intrinsic worldvolume theory is that of a $p+1$-dimensional fluid with a conserved q-brane number current, with $q \leq p$. The case $q = 0$ is a familiar one — a fluid with a conserved particle number — but fluids with string or brane currents do not seem to have been studied before, at least not relativistic ones[5]. In this paper we develop in detail the general theory of perfect fluids with q-brane currents for $q = 0, 1$. Blackfolds with $q = p$ (plus other examples) will be developed in [17] and the study of generic q will be presented elsewhere.

We perform an analysis of stationary solutions that completely solves the intrinsic equations in a very general and explicit manner, and which paves the way for the construction of new black holes with charges and dipoles. The extrinsic equations are in general differential equations, but, in analogy to [21], we find several classes of simple worldvolume geometries that can be solved algebraically. These are:

Charged black holes:

- Charged black holes with horizon topology S^{D-2} in $D \geq 6$, with rotation on s independent planes (with $1 \leq s < \frac{D-3}{2}$) and arbitrary dilaton coupling. These are the generalization of the Myers-Perry solutions to include electric charge, in the regime where their horizons are flattened along the planes of rotation (which can, but need not, be ultraspinning). In particular, they include the elusive electric rotating black holes of the Einstein-Maxwell theory ($a = 0$), which for the first time are described in these regimes. They also include the previously known charged rotating black holes in Kaluza-Klein theory [8, 9], whose properties we show are accurately reproduced in the appropriate regime. We shall argue that in the blackfold regime all these black holes are unstable even when they are close to extremality and slowly rotating.

- Charged black holes with horizon topology $\Pi_{p_i \in \mathrm{odd}} S^{p_i} \times S^{D-\sum_i p_i - 2}$. These are the natural charged generalization of the solutions in [18, 21]. Besides the exact five-dimensional black rings of [5, 6, 7], which we reproduce correctly when the rings are thin, no other construction of them exists.

String-dipole black holes:

[5]Liquid crystals are examples of fluids with string-like excitations, but we have not found any useful way to capitalize on their vast literature.

- Black holes with string dipole in $D \geq 5$, with horizon topology $\Pi_{p_i \in \text{odd}} S^{p_i} \times S^{D-\sum_i p_i - 2}$. The properties of the exact five-dimensional dipole rings of [13] are accurately reproduced in the ultraspinning limit.
- A qualitatively different class of black rings with string dipole in $D \geq 6$. They are constructed as blackfolds with annulus-shaped ($S^1 \times I$) and more generally solid-ring-shaped ($S^1 \times B_{2k-1}$) worldvolumes. Their horizon topology is ring-like, $S^1 \times S^{D-3}$, but their geometry is unlike the previous solutions: the S^{D-3} is pancaked along the planes on which the strings lie, so the shape resembles that of a 'prolate ring'[6]. There are also blackfolds with worldvolumes $I \times S^{2k-1}$ which give rise to similarly prolate versions of dipole black holes with product-of-spheres horizons.

This is far from being an exhaustive classification, instead it is mainly intended to illustrate the power of the method, in particular, of the general solution of the intrinsic equations, and the new regimes near extremality that we can also investigate.

Finally, it must be noted that although we have written the theories (1.1) without Chern-Simons terms, which are present in particular in five-dimensional minimal supergravity, in many cases the solutions we construct also apply with them. When the rotation occurs in a direction parallel to the brane current, as *e.g.,* in dipole black strings and rings, it generates no magnetic field and the Chern-Simons term is irrelevant (dipole black rings in five dimensions are often regarded as magnetic solutions of the $q = 0$ theory (1.1), and in this case it is the absence of induced electric sources that makes the Chern-Simons term unimportant to them).

When the rotation occurs in directions transverse to the current, as is necessarily the case for black holes with electric 0-brane charges, and possibly when $q < p$, then the rotation induces a magnetic dipole moment and then it makes a difference whether the Chern-Simons term is present or not. Even if at first sight this effect might seem subleading in the blackfold expansion, it is known that in the case of five-dimensional black rings it leads to rotation in the S^2 and to a regularity condition of global type, namely absence of Dirac-Misner strings, which requires keeping both the electric charge and the magnetic dipole even at arbitrarily large, but finite, ring radius. This is feasible in the blackfold framework, but is outside the scope of this article.

Outline of the paper. Section 2 develops the general formalism for blackfolds with q-brane charges. Although the main focus is on $q = 0, 1$, we obtain the effective stress-energy tensor for black p-branes with generic q-brane charges. In section 3 we solve in a complete and explicit manner the intrinsic, fluid-dynamical equations for stationary configurations with $q = 0, 1$. We present a simple action principle for the extrinsic dynamics, give expressions for the physical properties of the solutions, and obtain their thermodynamics. In section 4 we discuss the applicability of the method and identify the new regime of near-extremal, slowly rotating charged blackfolds. The

[6] We call a shape *prolate* if the directions in which it is lengthened include directions orthogonal to the rotation planes (which are typically lengthened too). This effect, induced by the repulsion among parallel strings, is to be contrasted to the flattening due to centrifugal forces.

subsequent sections are devoted to the construction and study of specific solutions obtained by applying the general study of sec. 3. Sec. 5 focuses on blackfolds with 0-brane charge, first with disk and even-ball worldvolumes, then with odd-sphere worldvolumes. Sec. 6 analyzes blackfolds with string dipole, and in particular we find that disk solutions are not possible, instead a new class of rings appears. We conclude in section 7 with a discussion of our results.

We collect a number of technical details in a series of appendices. Appendix A constructs the black p-brane solutions of (1.1) with q-brane charge as uplifts of Gibbons-Maeda black holes, and appendix B computes their stress-energy tensor and all physical magnitudes of interest, and discusses their thermodynamics. Appendix C compares the ultraspinning regime of the exact solutions for rotating black holes with Kaluza-Klein electric charge [9], and for black strings and rings with string charge [13] with the solutions in secs. 5 and 6. In all cases the agreement is perfect.

Notation. We follow ref. [19]:

Spacetime (background) objects: coordinates x^μ, metric $g_{\mu\nu}$, connection ∇_μ, with $\mu,\nu,\rho,\dots = 0,\dots,D-1$; $h_{\mu\nu}$ is the first fundamental form of the embedding of the worldvolume $\mathcal{W}_{p+1}$ in the background; $\perp^\mu{}_\nu$ is the projector orthogonal to $\mathcal{W}_{p+1}$.

Worldvolume objects: coordinates σ^a, with $a,b,c,\dots = 0,\dots,p$; the metric γ_{ab} induced on the worldvolume is the pullback of $h_{\mu\nu}$; its compatible connection, D_a, is the pullback of $h_\mu{}^\nu\nabla_\nu$.

Worldvolume current objects are denoted with hats: $\hat{h}_{ab}$ is the first fundamental form of the current worldline/sheet $\mathcal{C}_{q+1}$ inside $\mathcal{W}_{p+1}$; $\hat{\perp}_{ab} = \gamma_{ab} - \hat{h}_{ab}$ the projector orthogonal to $\mathcal{C}_{q+1}$; $|\hat{h}|^{1/2}$ is the area element of $\mathcal{C}_{q+1}$.

For a p-brane in D spacetime dimensions we define

$$n = D - p - 3 \geq 1\,. \tag{1.2}$$

Instead of the dilaton coupling a it is more convenient to use N such that

$$a^2 = \frac{4}{N} - \frac{2(q+1)(D-3-q)}{D-2}\,. \tag{1.3}$$

N is preserved under dimensional reduction and is a real number in general, but in many instances of interest in supergravity and string theory it is an integer.

When we talk about black holes, as opposed to infinitely extended black branes, we find more appropriate to use the term 'charged black hole' for the solutions with 0-brane charge, and 'dipole black holes' for the ones with $q \geq 1$ brane dipole.

2 Blackfolds with electric charge and with string dipoles

Refs. [18, 19] show that whenever there are two separate scales along the horizon of a black hole, one can effectively describe its physics by integrating out the smaller scale, leading to a thin black p-brane curved on a submanifold $\mathcal{W}_{p+1}$ embedded in a background spacetime. Its dynamics is

determined by a stress tensor $T_{\mu\nu}$ that is supported on its worldvolume. The geometry of the submanifold $\mathcal{W}_{p+1}$ is completely defined by the combination of its induced worldvolume metric

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{2.1}$$

that fixes the intrinsic geometry, and of the extrinsic curvature tensor $K_{\mu\nu}{}^\rho$ that encodes the shape of the embedding. The projector to the tangent space of $\mathcal{W}_{p+1}$ is given by the first fundamental form

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab}, \tag{2.2}$$

while the projector to orthogonal directions is

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \,. \tag{2.3}$$

Then $K_{\mu\nu}{}^\rho = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\sigma h_\lambda{}^\rho$, and its trace is the mean curvature vector $K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho$.

The collective variables describing a neutral blackfold are the embedding functions $X^\mu(\sigma)$ characterizing the manifold $\mathcal{W}_{p+1}$, the thickness $r_0(\sigma)$ of the horizon, and the local boost field $u^a(\sigma)$ satisfying $\gamma_{ab} u^a u^b = -1$. All collective variables depend on the worldvolume coordinates σ^a, $a = 0, \ldots, p$, and are allowed to vary over a length scale R much longer than the scale r_0, set by the brane thickness, of the physics we have integrated out (we will be more precise about this point in sec. 4). The result is a long-wavelength effective theory of the black hole defined by the effective stress tensor $T_{\mu\nu}$ of the brane.

When the black hole is charged, there are additional collective variables that combine into an effective charge current J that flows on the blackfold worldvolume $\mathcal{W}_{p+1}$. In general this will be a $(q+1)$-form describing a conserved q-brane current in $\mathcal{W}_{p+1}$.[7] The collective variables that characterize it are a q-brane charge density $\mathcal{Q}(\sigma)$ and a set of worldvolume vectors for the orientation (polarization) of the q-branes inside the worldvolume. The applicability of the blackfold approach in the presence of charges will be discussed later.

2.1 Fluid dynamics with particle- and string-number currents

In this article we restrict ourselves to blackfolds which carry 0-brane (particle) and 1-brane (string) charges diluted in their worldvolume. The formalism for perfect fluids with a conserved q-brane current with $q = 0, 1$, which we develop next, remains simple enough.

When $q = 0$ this is the familiar fluid dynamics with a conserved particle number (like *e.g.,* baryon number). The stress-energy tensor of the fluid, T_{ab}, has a unique unit-normalized timelike eigenvector u whose eigenvalue $-\varepsilon$ defines the energy density. Spatial isotropy then implies that

$$T_{ab} = \varepsilon u_a u_b + P(\gamma_{ab} + u_a u_b) \tag{2.4}$$

[7] Note that there is no gauge field on the worldvolume theory, and the worldvolume current J is a global one. This is just like in the AdS/CFT correspondence, with which the blackfold approach has many similarities.

with pressure P. As is well known, in the absence of dissipative effects (specifically, no particle diffusion) the particle current must be proportional to u,

$$J_a = \mathcal{Q}\, u_a \,, \tag{2.5}$$

where $\mathcal{Q}$ is the charge density on the fluid.

When $q = 1$, we can contract the timelike eigenvector u of T_{ab} with the string two-form current J_{ab} to find the spacelike vector $u_a J^{ab}$, which is orthogonal to u. Normalizing this vector v to one, so that

$$-\, u^2 = v^2 = 1 \,, \qquad u \cdot v = 0 \,, \tag{2.6}$$

we can then write

$$J_{ab} = \mathcal{Q}(u_a v_b - v_a u_b) \,, \tag{2.7}$$

which defines the string charge density $\mathcal{Q}$.

The vector v breaks the spatial isotropy of the fluid and characterizes the directions along which the (dissolved) strings lie. Again, if dissipative currents are absent, v must be an eigenvector of T_{ab}. The fluid is isotropic in the spatial directions transverse to it, so

$$T_{ab} = \varepsilon u_a u_b + P_{\parallel} v_a v_b + P_{\perp} \left(\gamma_{ab} + u_a u_b - v_a v_b\right) \,. \tag{2.8}$$

Note that this analysis easily allows the simultaneous presence of 0- and 1-brane-number currents, but in this paper we will not study such systems.

We can unify the description of fluids with either of these currents if we introduce the projector onto the space parallel to the particle/string worldline/sheet,

$$\hat{h}_{ab} = -u_a u_b + q\, v_a v_b \qquad (q = 0, 1) \,, \tag{2.9}$$

and onto directions in $\mathcal{W}_{p+1}$ orthogonal to it,

$$\hat{\perp}_{ab} = \gamma_{ab} - \hat{h}_{ab} \,. \tag{2.10}$$

Then

$$T_{ab} = (\varepsilon + P_{\parallel}) u_a u_b + (P_{\parallel} - P_{\perp}) \hat{h}_{ab} + P_{\perp} \gamma_{ab} \,. \tag{2.11}$$

If we also introduce the volume form $\hat{V}$ on the worldline/sheet,

$$\hat{V} = \begin{cases} u & \text{for } q = 0 \\ u \wedge v & \text{for } q = 1 \,, \end{cases} \tag{2.12}$$

then

$$J = \mathcal{Q}\, \hat{V} \,. \tag{2.13}$$

The difference between the pressure in directions orthogonal to the strings and parallel to them is due to the energy density of the strings (essentially their tension), given by $\Phi \mathcal{Q}$, with Φ the string chemical potential, so in general

$$P_{\perp} - P_{\parallel} = \Phi \mathcal{Q} \,. \tag{2.14}$$

In addition to this, thermodynamic equilibrium is satisfied locally on the worldvolume so we have the first law

$$d\varepsilon = \mathcal{T} ds + \Phi d\mathcal{Q}\,, \tag{2.15}$$

where $\mathcal{T}$ is the local temperature, and also the thermodynamic Gibbs-Duhem relations

$$\varepsilon + P_\perp = \mathcal{T} s + \Phi \mathcal{Q}\,, \tag{2.16}$$

$$dP_\perp = s d\mathcal{T} + \mathcal{Q} d\Phi\,, \qquad dP_\parallel = s d\mathcal{T} - \Phi d\mathcal{Q}\,. \tag{2.17}$$

Counting the independent collective variables, we find p independent components for u, $p-1$ components for v when $q=1$, plus the energy density, charge density, and the pressures, which add up to $(q+1)p+3$ variables. They must satisfy the current continuity equations

$$d * J = 0 \tag{2.18}$$

($*$ is the Hodge dual on $\mathcal{W}_{p+1}$) and the fluid equations

$$D_a T^{ab} = 0\,. \tag{2.19}$$

This is a set of $(q+1)p+2$ equations, which, when supplemented with the equation of state that determines ε for given pressures and charge density, are enough to determine the system.

We proceed now to write down these equations more explicitly. We denote

$$\dot{u} \equiv u^a D_a u\,, \qquad D_v v \equiv v^a D_a v\,. \tag{2.20}$$

Intrinsic equations with particle and string currents. The exterior product of the current continuity equation (2.18) with $*\hat{V}$ gives

$$* \hat{V} \wedge d * \hat{V} = 0\,. \tag{2.21}$$

This equation expresses, through Frobenius' theorem, the existence of integral $(q+1)$-dimensional submanifolds parallel to $\hat{h}_{ab}$. This is trivial for particle currents, but for string currents it implies the integrability of the submanifolds $\mathcal{C}_2 \subset \mathcal{W}_{p+1}$ whose tangent space is spanned by (u,v). These submanifolds are the worldsheets of the strings carrying the charge $\mathcal{Q}$. In index notation the equation is

$$\hat{\perp}_{ab}\, [u,v]^b = 0\,. \tag{2.22}$$

The remaining current continuity equations can be written as

$$D_a(\mathcal{Q} u^a) = 0 \tag{2.23}$$

for $q=0$ and

$$D_a(\mathcal{Q} u^a) + \mathcal{Q} u^a D_v v_a = 0\,, \tag{2.24}$$

$$D_a(\mathcal{Q} v^a) - \mathcal{Q} v^a \dot{u}_a = 0\,, \tag{2.25}$$

for $q=1$. Using these and the local thermodynamics equations we can write the fluid equations (2.19) as the conservation of entropy

$$D_a(su^a) = 0\,, \tag{2.26}$$

and the (Euler) force equations

$$\hat{\perp}^{ab} s\mathcal{T}\left(\dot{u}_b + \partial_b \ln \mathcal{T}\right) - \mathcal{Q}\Phi\left(\hat{K}^a - \hat{\perp}^{ab}\partial_b \ln \Phi\right) = 0 \tag{2.27}$$

and

$$\left(\hat{h}^{ab} + u^a u^b\right)\left(\dot{u}_b + \partial_b \ln \mathcal{T}\right) = 0 \tag{2.28}$$

(which is trivial when $q=0$). Here we have introduced the mean curvature of the worldlines/sheets embedded in $\mathcal{W}_{p+1}$,

$$\hat{K}^a = \hat{h}^{bc} D_b \hat{h}_c{}^a = -\hat{\perp}^a{}_b\left(\dot{u}^b - q\, D_v v^b\right) \qquad (q=0,1)\,. \tag{2.29}$$

2.2 Extrinsic dynamics

The theory of blackfolds has a second set of equations, extrinsic ones, that describe the dynamics of the embedding of the p-brane worldvolume in the D-dimensional background spacetime. The $D-p-1$ independent transverse coordinates $X(\sigma)$ that characterize this embedding are determined by solving Carter's equations [22],

$$T^{\mu\nu} K_{\mu\nu}{}^\rho = \frac{1}{(q+1)!}\perp^\rho{}_\sigma J_{\mu_0\ldots\mu_q} H^{\mu_0\ldots\mu_q\sigma}\,, \tag{2.30}$$

where $H_{[q+2]}$ is the background field strength that couples electrically to the q-brane charge. In this article we shall be concerned only with situations where such fields are absent, so

$$T^{\mu\nu} K_{\mu\nu}{}^\rho = 0\,. \tag{2.31}$$

The stress tensor is the push-forward $T^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu T^{ab}$ of the q-brane-charged fluid stress tensor (2.11). Then (2.31) becomes (*e.g.*, see eq. (2.2) of [21])

$$P_\perp K^\rho = -\perp^\rho{}_\mu\left(s\mathcal{T}\dot{u}^\mu - \Phi\mathcal{Q}\hat{K}^\mu\right)\,, \tag{2.32}$$

where

$$\hat{K}^\mu = \hat{h}^{\nu\sigma}\nabla_\nu \hat{h}_\sigma{}^\mu = \hat{h}^{\nu\sigma} K_{\nu\sigma}{}^\mu \tag{2.33}$$

is the mean curvature of the worldlines/sheets of the particles/strings embedded in the D-dimensional background spacetime, with the first fundamental form $\hat{h}_{\mu\nu}$ obtained by pushing forward (2.9). Note that $\hat{K}^\mu$ has components both along transverse and along parallel directions to $\mathcal{W}_{p+1}$, the pull-back of the latter being (2.29).

2.3 Effective stress-energy tensor for charged blackfolds

The next step is to compute the effective stress tensor for a black p-brane with q-brane charge in its worldvolume. The expressions we find in this section are valid for $0 \leq q \leq p$, although in this paper we only put them to use when $q = 0, 1$.

The effective stress tensor of the blackfold is defined as the quasi-local stress tensor of Brown and York [23] measured at a large distance in directions transverse to the brane. We will restrict ourselves to the leading order in a derivative expansion of T^{ab}, *i.e.,* a perfect fluid. Dissipative corrections can be computed in the blackfold formalism [24], but they do not alter the properties of the stationary blackfolds that are the main interest of this article.

To this leading order we must compute the quasi-local stress tensor for the corresponding planar, equilibrium brane. A large class of exact solutions of the theories (1.1), describing black p-branes with diluted q-brane charges in D-dimensional general relativity coupled to a $(q+1)$-form gauge potential $B_{[q+1]}$ and a dilaton ϕ with dilaton coupling a, is obtained in appendix A, see eqs. (A.25)–(A.29). They depend on two parameters, the radius r_0 and a charge parameter α. The quasi-local stress tensor they source is computed in appendix B.1. It is an anisotropic perfect fluid, since the diluted q-brane charges sort out privileged directions along which their worldvolumes extend. Indeed, the net effect of these additional branes is to increase the tension along the directions parallel to them. The energy density ε, the pressure $P_\parallel$ in the q-brane directions and the pressure $P_\perp$ in the transverse directions are found to be

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + nN \sinh^2 \alpha\right), \tag{2.34}$$

$$P_\parallel = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1 + nN \sinh^2 \alpha\right), \qquad P_\perp = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n. \tag{2.35}$$

Here, α is the charge parameter of the solution, linked to the electric potential

$$\Phi = \sqrt{N} \tanh \alpha\,, \tag{2.36}$$

while the charge density, computed in appendix B.2, is

$$\mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \sinh \alpha \cosh \alpha\,. \tag{2.37}$$

The parameter N was defined in (1.3) (see also (A.27)). Observe that all these densities depend on n and N, but not on p nor q. The reason is that under compactification of the p- and q-brane directions these densities become the conserved charges of a dilatonic black hole, which depend only on the number $n + 3$ of dimensions that the black hole lives in, and on the dilaton coupling through the parameter N that is invariant under dimensional reduction.

From these expressions we find the equation of state

$$\varepsilon = -P_\parallel - nP_\perp\,, \tag{2.38}$$

or using (2.14), which is also satisfied, we can write it as

$$\varepsilon - \Phi\mathcal{Q} = -(n+1)P_\perp\,. \tag{2.39}$$

Finally, the local temperature and entropy density of this black brane are

$$\mathcal{T} = \frac{n}{4\pi r_0 \left(\cosh\alpha\right)^N}\,, \qquad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \left(\cosh\alpha\right)^N . \tag{2.40}$$

In the blackfold framework one promotes the constants r_0 and α defining the black brane solution to slowly varying functions $r_0(\sigma)$ and $\alpha(\sigma)$ of the worldvolume coordinates. Furthermore, a boost along the worldvolume is introduced, which is taken to be characterized by a slowly varying velocity $u^a(\sigma)$. For string charge, the polarization vector becomes another collective variable, $v^a(\sigma)$. A solution for the fluid is fully specified when we determine $u(\sigma)$ and $v(\sigma)$, and a pair of worldvolume functions such as $(\mathcal{T}(\sigma), \Phi(\sigma))$ or $(r_0(\sigma), \alpha(\sigma))$.

For the specific blackfold fluid given by (2.34), (2.35), the stress tensor (2.11) is

$$T_{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n\, u_a u_b - \gamma_{ab} - nN \sinh^2\alpha\; \hat{h}_{ab} \right) \tag{2.41}$$

and the extrinsic equation becomes

$$K^\rho = n \perp^\rho{}_\mu \left(\dot{u}^\mu - N \sinh^2\alpha\; \hat{K}^\mu \right) . \tag{2.42}$$

In particular, for 0-brane charge this is

$$K^\rho = n(1 + N\sinh^2\alpha) \perp^\rho{}_\mu \dot{u}^\mu . \tag{2.43}$$

We see that the effect of 0-brane charge dissolved on the worldvolume is to *decrease* the value of the acceleration needed to sustain a worldvolume of given mean curvature. This is because, for a given energy density, the 0-brane charge decreases the effective tension on the worldvolume of the black brane, thus making it easier to bend it. This has significant consequences that we discuss in sec. 4.

For blackfolds with q-brane charge with $q > 0$, the worldvolumes $\mathcal{C}_{q+1}$ with positive mean curvature have the effect of *increasing* the acceleration needed to sustain a given curvature of $\mathcal{W}_{p+1}$. This is in agreement with our previous observation that, for a given transverse tension, q-brane charge increases the tension along the directions in which the q-branes extend.

2.4 Blackfolds with boundaries

The equations that a charged blackfold must satisfy if its worldvolume has boundaries are a simple extension of those found in [19] for neutral blackfolds. If $f(\sigma)$ is a function that defines the boundaries by $f|_{\partial\mathcal{W}_{p+1}} = 0$, then we must have

$$u^\mu \partial_\mu f|_{\partial\mathcal{W}_{p+1}} = 0 \tag{2.44}$$

at all boundaries.

Additionally, the string current conservation equation requires

$$\mathcal{Q}\, u^{[\mu} v^{\nu]} \partial_\mu\, f|_{\partial\mathcal{W}_{p+1}} = 0 \tag{2.45}$$

so

$$\mathcal{Q}\, v^\mu \partial_\mu f|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{2.46}$$

i.e., the string current must remain parallel to the boundaries.

Conservation of the stress-energy requires that in the absence of any surface tension, the fluid pressure must vanish on the boundary,

$$P_\perp|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{2.47}$$

For a blackfold fluid this means that at the edge,

$$r_0|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{2.48}$$

It must be noted that this is a necessary condition for regularity of the blackfold horizon, but it is not sufficiently understood yet under what circumstances it may also be sufficient.

The meaning of (2.48) for charged blackfolds will be further clarified after we analyze stationary solutions.

3 Stationary charged blackfolds

The general study of charged stationary blackfolds can be performed following closely the lines of [19]. The intrinsic equations can be solved completely and then the extrinsic equations can be conveniently encoded in a simple variational principle. Explicit expressions for the physical magnitudes can be found which satisfy the laws of black hole thermodynamics.

3.1 Solving the intrinsic fluid equations

Stationarity implies that the black brane has a Killing horizon associated to a timelike Killing vector k with a surface gravity κ. The velocity field of the fluid is set to be proportional to this Killing vector,[8]

$$u^a = \gamma k^a\,, \qquad \gamma = 1/\sqrt{-k^2}\,. \tag{3.1}$$

This is a general result, proven in [25] for general stationary fluid configurations, and extended in [19] to neutral blackfolds. In the presence of charges, we assume again this form of the velocity field, which implies that

$$D_{(a}u_{b)} = u_{(a}\partial_{b)} \ln\gamma\,, \tag{3.2}$$

and therefore the expansion and shear of u vanish and its acceleration is

$$\dot{u}_a = -\partial_a \ln\gamma. \tag{3.3}$$

Stationarity of the full configuration requires that any physical quantity characterized by a tensor $\mathbb{T}$ be constant along the Killing flow, so its Lie derivative vanishes

$$\mathcal{L}_k \mathbb{T} = 0\,. \tag{3.4}$$

[8]Do not confuse the redshift factor γ with the metric induced in the worldvolume γ_{ab}.

Note that (3.1) implies $\mathcal{L}_k u = 0$ and $\mathcal{L}_k \gamma = 0$.

In order to solve the intrinsic equations for a charged fluid we use (3.3) and that, in general, the extrinsic curvature vector of the worldlines/sheets inside $\mathcal{W}_{p+1}$ is

$$\hat{K}^a = -\hat{\perp}^{ab}\partial_b \ln\left(|\hat{h}|^{1/2}\right) , \tag{3.5}$$

where $|\hat{h}|^{1/2}$ is the area element of the worldlines/sheets. Then (2.27) and (2.28) become

$$s\mathcal{T}\hat{\perp}^{ab}\partial_b \ln\frac{\mathcal{T}}{\gamma} + \mathcal{Q}\Phi\hat{\perp}^{ab}\partial_b \ln\left(|\hat{h}|^{1/2}\Phi\right) = 0 \tag{3.6}$$

and

$$\left(\hat{h}^{ab} + u^a u^b\right)\partial_b \ln\frac{\mathcal{T}}{\gamma} = 0 \, . \tag{3.7}$$

In principle these equations would allow a dependence of $\mathcal{T}$ and Φ on the coordinate along u (*i.e.*, time), but this would be incompatible with stationarity. We solve the equations by taking

$$\mathcal{T}(\sigma) = \gamma T \, , \tag{3.8}$$

where T is an integration constant with the interpretation of global temperature of the blackfold, and

$$\Phi(\sigma) = \frac{\phi(\sigma)}{|\hat{h}|^{1/2}} \, , \tag{3.9}$$

where $\phi(\sigma)$ can vary only along spatial directions parallel to the current but orthogonal to u,

$$(\hat{\perp}^{ab} - u^a u^b)\partial_b \phi(\sigma) = 0 \, . \tag{3.10}$$

Then if we integrate over the spatial directions along the current we find a constant

$$\Phi_H = \int_{\mathcal{C}_q} d^q\sigma \phi(\sigma) = \int_{\mathcal{C}_q} d^q\sigma |\hat{h}|^{1/2}\Phi(\sigma) \tag{3.11}$$

that may be regarded as the global potential.[9]

For 0-brane currents $\phi = \Phi_H$ fixes the local electric potential $\Phi(\sigma)$ on the blackfold. Thus, given the Killing vector k and the constants T and Φ_H all the collective variables for the fluid are determined. This is therefore a complete and general solution to the intrinsic equations for a stationary blackfold with 0-brane charge.

For the fluid with string-number current the solution is not complete yet, as we still need to find v that solves (2.22), and then fully specify Φ. In order to complete the solution we use the result [26] that for a stationary (but not static) black hole, there is a spacelike Killing vector ψ that commutes with k,

$$[\psi, k] = 0 \, . \tag{3.12}$$

[9] The necessity to integrate the potential along the worldvolume directions parallel to the current for obtaining the condition for stationarity has been recognized previously in [17].

Then we construct its component orthogonal to k

$$\zeta = \psi - \frac{\psi^a k_a}{k^2} k \tag{3.13}$$

and assume that ζ is spacelike over the blackfold worldvolume. If ζ were found to become timelike on a region of the worldvolume, then that region should be excluded and the blackfold would have a boundary.

Note that $\psi^a k_a / k^2$ is a function that remains constant along the two Killing directions ψ and k, but which can vary in directions transverse to them. So in general ζ is not a Killing vector, but nevertheless it satisfies

$$\zeta^a D_{(a}\zeta_{b)} = 0\,, \qquad k^a D_{(a}\zeta_{b)} = 0\,, \qquad D_a \zeta^a = 0\,. \tag{3.14}$$

If we take

$$v^a = \frac{\zeta^a}{|\zeta|}\,, \tag{3.15}$$

then

$$[u, v] = 0 \tag{3.16}$$

so eq. (2.22) is satisfied. Additionally,

$$D_a v^a = 0\,, \qquad D_v v_a = -\partial_a \ln|\zeta|\,, \qquad v^a \dot{u}_a = 0 \tag{3.17}$$

and eq. (3.2) implies that $v^a D_v u_a = 0$, or equivalently, $u^a D_v v_a = 0$. With these (2.25) becomes

$$v^a \partial_a \mathcal{Q} = 0\,. \tag{3.18}$$

Together with (3.8) this implies that Φ cannot vary along v, so in (3.9) we set ϕ to be a constant. If we assume that the orbits of ψ are closed we can normalize it so that the periodicity is 2π. The integration in (3.11) only involves the cyclic direction along ψ, so $\Phi_H = 2\pi\phi$.

The area element on $\mathcal{C}_{q+1}$ is

$$|\hat{h}|^{1/2} = \begin{cases} \gamma^{-1} & \text{for } q = 0 \\ |\zeta||k| = |\zeta|/\gamma & \text{for } q = 1\,, \end{cases} \tag{3.19}$$

and then we can write the solution for the potential, both for $q = 0, 1$, as

$$\Phi(\sigma) = \frac{1}{(2\pi)^q} \frac{\Phi_H}{|\hat{h}(\sigma)|^{1/2}}\,. \tag{3.20}$$

Inverting (2.36) and (2.40) we find

$$r_0(\sigma) = \frac{n}{4\pi T}\gamma^{-1}\left(1 - \frac{1}{(2\pi)^{2q}} \frac{\Phi_H^2}{N|\hat{h}|}\right)^{N/2}\,, \qquad \tanh\alpha(\sigma) = \frac{1}{(2\pi)^q} \frac{\Phi_H}{\sqrt{N}|\hat{h}|^{1/2}}\,. \tag{3.21}$$

It is now clear that the equations for s and $\mathcal{Q}$ are all solved. Stationarity of all physical quantities is also guaranteed.

For $q=0$ this solution is general, while for $q=1$ the only stationary solutions that are not included in this analysis are such that the spatial direction of the current is not aligned with a vector constructed out of Killing vectors in the form (3.13) (possibly because ψ may not exist, in which case the blackfold is necessarily static). In this case they must still satisfy (3.8) and (3.9), but we do not have a complete explicit form for v or Φ. Such solutions may be of interest but we will not investigate them in this paper.

Observe that our general stationary solution (3.9)

$$\hat{K}^a = \hat{\perp}^{ab}\partial_b \ln \Phi \,, \tag{3.22}$$

says that the gradient of the chemical potential balances the stress due to the mean curvature of the dissolved q-branes. When $q=0$ the latter only involves the acceleration of the charged particles, but when $q=1$ there is also a contribution (typically dominant) from the spatial curvature of the string worldsheets. This equation can be recovered by extremizing an action for the embedding of $\mathcal{C}_{q+1}$ in the blackfold worldvolume,

$$\hat{I} = \int_{\mathcal{C}_{q+1}} d\hat{V}_{(q+1)}\, \Phi \,, \tag{3.23}$$

where $d\hat{V}_{(q+1)}$ is the measure on $\mathcal{C}_{q+1}$ and the variations deform the embedding of $\mathcal{C}_{q+1}$ inside $\mathcal{W}_{p+1}$. Using our solution (3.9), (3.11), we can write this as the action for a point particle of effective mass Φ_H that moves along the worldvolume directions transverse the current,

$$\hat{I} = \Phi_H \int d\tau \tag{3.24}$$

where τ is the proper time along the trajectory of the "particle".

3.2 Extremal limit at the blackfold boundary

For worldvolumes with boundaries the presence of charges (or dipoles) introduces a novelty compared to neutral blackfolds.

We have determined in (2.48) that r_0 must vanish at the boundary of a blackfold, and for a stationary solution r_0 is given by (3.21). Then, for a neutral blackfold we must have $\gamma \to \infty$, *i.e.,* k becomes null at the boundary, which may happen because the blackfold meets a Killing horizon of the background, or because the fluid velocity locally reaches the speed of light. In the latter case, it must be noted that even if the lightlike-boost limit of a *uniform* neutral black brane results in a naked singular solution, for a blackfold this is a phenomenon localized at the boundary of the worldvolume and the full horizon of the blackfold can still be regular, as is the case with the disks of [18, 19] and those discussed below.

When charges are present r_0 can vanish not only where k becomes a null vector, but also at other boundaries where, for a given Φ_H,

$$|\hat{h}|^{1/2} = \frac{1}{(2\pi)^q}\frac{\Phi_H}{\sqrt{N}} \,. \tag{3.25}$$

Then k can remain timelike, but the charge boost α diverges: the black brane becomes (locally) *extremal.* Instances of this phenomenon will appear in secs. 5.1 and 6.1.

When T is finite and non-zero this extremal limit will occur only in a localized manner, at some of the boundaries of the blackfold but not away from them. Therefore even if the extremal limit of the uniform brane may be singular, this need not imply a singularity of the blackfold horizon and indeed in secs. 5.1 and 6.1 we present evidence of this.

3.3 Extrinsic equations and action for stationary blackfolds

We make the natural assumption that the vector fields k and ψ of the previous analysis are the pullbacks to the worldvolume of commuting Killing vectors of the background spacetime. Thus equations (3.2), (3.3) and (3.17) are also satisfied as equations in the background. The components of $\hat{K}^\mu$ in directions orthogonal to the worldvolume, which enter in the extrinsic equations (2.42), can now be obtained from

$$\hat{K}^\mu = -\hat{\perp}^{\mu\nu}\partial_\nu \ln\left(|\hat{h}|^{1/2}\right) \tag{3.26}$$

with $|\hat{h}|^{1/2}$ as in (3.19) and $\hat{\perp}^{\mu\nu} = g^{\mu\nu} - \hat{h}^{\mu\nu}$.

Using the intrinsic worldvolume solutions to the stationarity conditions, the extrinsic equations (2.42) reduce to

$$\begin{aligned} K^\rho &= n \perp^{\rho\mu} \partial_\mu \ln r_0 \\ &= \perp^{\rho\mu} \partial_\mu \ln(-P_\perp)\,, \end{aligned} \tag{3.27}$$

both for $q = 0, 1$.

Applying the results of [19], this equation can be equivalently found by varying, under deformations of the brane embedding, the action

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-h}\,P_\perp\,. \tag{3.28}$$

The form of this action is a familiar one for p-branes (*e.g.,* for Dirac branes, where the tension $-P$ is uniform), and for thermodynamic systems in equilibrium (if we recall that $-P_\perp$ is equal to the Gibbs free energy density). It is remarkable that the resulting action depends explicitly only on the thickness r_0 of the blackfold (and the embedding coordinates), regardless of the presence of charges and anisotropy.

Assume now that the background spacetime has a timelike Killing vector ξ, canonically normalized to generate unit time translations at asymptotic infinity, and whose norm on the worldvolume is

$$-\,\xi^2|_{\mathcal{W}_{p+1}} = R_0^2(\sigma)\,. \tag{3.29}$$

Let us further assume that ξ is hypersurface-orthogonal, so we can foliate the blackfold in spacelike slices $\mathcal{B}_p$ normal to ξ. The unit normal to $\mathcal{B}_p$ is

$$n^a = \frac{1}{R_0}\xi^a\,. \tag{3.30}$$

Thus R_0 measures the (gravitational) redshift between worldvolume time and asymptotic time. When integrating over $\mathcal{W}_{p+1}$ we can split the trivial integration over the Killing time generated by ξ, which gives an overall factor β of the time interval, so

$$I = \frac{\beta\Omega_{(n+1)}}{16\pi G}\int_{\mathcal{B}_p} dV_{(p)}\, R_0 r_0^n, \tag{3.31}$$

where $dV_{(p)}$ is the integration measure on $\mathcal{B}_p$.

This result immediately implies the condition eq. (2.48) at the blackfold boundary $\partial\mathcal{B}_p$: the variations of the action that deform the boundary give $\delta_{\partial\mathcal{B}_p} I \propto r_0$, so the thickness r_0 must vanish at it.

3.4 Physical properties of stationary blackfolds and first law

Once the intrinsic fluid equations are solved we can proceed with the computation of the resulting thermodynamic quantities of the blackfold. Here we extend and improve the analyses in [19, 21].

Mass, angular momentum, entropy and charge. Let the stationarity Killing vector k^μ be given by a linear combination of orthogonal commuting Killing vectors of the background spacetime,

$$k^\mu = \xi^\mu + \sum_i \Omega_i \chi_i^\mu, \tag{3.32}$$

where ξ is the generator of time-translations of the background that we introduced above, and χ_i are generators of angular rotations in the background spacetime normalized such that their orbits have periods 2π. The angular velocities of the blackfold along these directions are then given by Ω_i. According to (3.1), (3.30) and (3.32),

$$\gamma = -\frac{n^a u_a}{R_0}\,. \tag{3.33}$$

Since $-n^a u_a$ is the gamma-factor for relativistic time-dilation in the fluid, we see that γ accounts for the redshifts caused by both gravitational and local Lorentz-boost effects.

The mass and angular momenta are given by the integrals of the corresponding densities over the worldvolume section $\mathcal{B}_p$,

$$M = \int_{\mathcal{B}_p} dV_{(p)}\, T_{ab} n^a \xi^b\,, \qquad J_i = -\int_{\mathcal{B}_p} dV_{(p)}\, T_{ab} n^a \chi_i^b\,. \tag{3.34}$$

The total entropy is deduced from the entropy current $s^a = s(\sigma) u^a$,

$$S = -\int_{\mathcal{B}_p} dV_{(p)}\, s_a n^a = \int_{\mathcal{B}_p} dV_{(p)}\, R_0 \gamma s(\sigma)\,, \tag{3.35}$$

and similarly, the 0-brane charge is

$$Q = -\int_{\mathcal{B}_p} dV_{(p)}\, J_a n^a = \int_{\mathcal{B}_p} dV_{(p)}\, R_0 \gamma \mathcal{Q}(\sigma) \qquad (q=0)\,. \tag{3.36}$$

The string charge is obtained by integrating the charge density over the directions $\mathcal{B}^{\perp}_{p-1}$ in $\mathcal{B}_p$ that are orthogonal to the current,

$$Q = -\int_{\mathcal{B}^{\perp}_{p-1}} dV_{(p-1)}\, J_{ab} n^a m^b , \qquad (q = 1) \tag{3.37}$$

where m is the unit spatial vector in $\mathcal{C}_2$ that is orthogonal to n. We can find that

$$v = \frac{m + (m_a u^a) u}{\sqrt{1 + (m_a u^a)^2}} \tag{3.38}$$

by demanding that it lies along $\mathcal{C}_2$ and satisfies (2.6). Then

$$J_{ab} n^a m^b = -R_0 \gamma_\perp \mathcal{Q} \,, \tag{3.39}$$

where

$$\gamma_\perp = \frac{\gamma}{\sqrt{1 + (m_a u^a)^2}} \tag{3.40}$$

can easily be seen to have the interpretation of the Lorentz contraction factor due to fluid velocity in directions transverse to the string current (times the gravitational redshift). When the rotation is parallel to the strings, then $\gamma_\perp = 1/R_0$. Observe that here we do not require the existence of a second Killing vector ψ, but if it exists, then $m = \psi/|\psi|$.

Along the spatial directions of the current we have

$$d^q\sigma \, |\hat{h}|^{1/2} = \frac{1}{\gamma_\perp} d\hat{V}_{(q)} \tag{3.41}$$

where $d\hat{V}_{(q)}$ is the integration measure on $\mathcal{C}_q$, along the direction of m. Then the global potential in (3.11) can be written

$$\Phi_H = \int_{\mathcal{C}_q} d\hat{V}_{(q)} \frac{\Phi(\sigma)}{\gamma_\perp} \tag{3.42}$$

where for $q = 0$ we recover $\gamma_\perp = \gamma$ and there is no integral to perform. This is valid for generic stationary blackfolds, including those for which the solution (3.15) does not apply. If it does, like in the examples we consider later where $m = \psi/|\psi|$, then $\gamma_\perp$ can be taken out of the integral.

Thus

$$\Phi_H Q = \int_{\mathcal{C}_q} d\hat{V}_{(q)} \int_{\mathcal{B}^{\perp}_{p-q}} dV_{(p-q)} \Phi R_0 \mathcal{Q} = \int_{\mathcal{B}_p} dV_{(p)} R_0 \Phi \mathcal{Q} \,. \tag{3.43}$$

Observe that, while the charge density $\mathcal{Q}$ has dimensions that depend only on n but not on q, and the local chemical potential Φ is dimensionless for any q, the integrated charge and potential have different dimensions for $q = 0$ and $q = 1$,

$$[GQ] = (\text{Length})^{D-3-q} \,, \qquad [\Phi_H] = (\text{Length})^q \tag{3.44}$$

(G is Newton's constant).

Thermodynamic relations and first law. Let us now express the action (3.31) in terms of these quantities. Begin by writing (3.28) as

$$I = \beta \int_{\mathcal{B}_p} dV_{(p)} n^a k_a P_\perp \tag{3.45}$$

(since $n_a k^a = -R_0$), and use the generic thermodynamic relations (2.14) and (2.16) in the stress tensor (2.11) to find that

$$\begin{aligned} k_a P_\perp &= T_{ab} k^b + \mathcal{T} s k_a + \Phi \mathcal{Q} k_a \\ &= T_{ab} \xi^b + \sum_i \Omega_i T_{ab} \chi_i^b + T s_a + \Phi \mathcal{Q} k_a \,. \end{aligned} \tag{3.46}$$

We have used (3.1), (3.8) and (3.32) here. Contracting with n^a and integrating over $\mathcal{B}_p$ we find

$$I = \beta W = \beta \left(M - TS - \sum_i \Omega_i J_i - \Phi_H Q \right) , \tag{3.47}$$

so we recover the identification between the action and the thermodynamic grand canonical potential W. Note that we have only used the intrinsic equilibrium solution and have not imposed the extrinsic equations. Actually, this derivation of (3.47) applies to any stationary charged fluid, since we have only used the generic relations (2.14), (2.16), but not the specific equation of state of the blackfold fluid.

In a sense, (3.47) is the integrated version of the local thermodynamic equation (2.16), so we should expect to have two more relations, from (2.14) and the equation of state (2.38). It is easy to show from these that

$$-P_\perp = \frac{1}{n} \mathcal{T} s \tag{3.48}$$

which upon integration, and with $\beta = 1/T$, gives

$$I = \frac{1}{n} S \,. \tag{3.49}$$

The remaining relation can be obtained through similar manipulations, in the form

$$(D-3)M - (D-2) \left(TS + \sum_i \Omega_i J_i \right) - (D-3-q) \Phi_H Q = \mathcal{T}_{\rm tot} \,, \tag{3.50}$$

where

$$\mathcal{T}_{\rm tot} = -\int_{\mathcal{B}_p} dV_{(p)}\, R_0 \left(\gamma^{ab} + n^a n^b \right) T_{ab} \tag{3.51}$$

is the total tensional energy, obtained by integrating the local tension over the blackfold volume [27, 28].

The Smarr relation for charged black holes,

$$(D-3)M - (D-2) \left(TS + \sum_i \Omega_i J_i \right) - (D-3-q) \Phi_H Q = 0 \tag{3.52}$$

must be recovered when one considers a Minkowski background, where $R_0 = 1$, and furthermore the extrinsic equations for equilibrium are satisfied. Thus, extrinsic equilibrium in Minkowski backgrounds must imply

$$\mathcal{T}_{\rm tot} = 0\,. \tag{3.53}$$

In simple instances where Carter's equations reduce to a single equation, one can simply impose this condition to derive the solution to the extrinsic equations. If the tensional energy did not vanish, it would imply the presence of sources of tension acting on the blackfold, *e.g.,* in the form of conical or stronger singularities of the background space. It should be interesting to derive (3.53) as a consequence of (3.27). Solutions with $R_0 \neq 1$, which do not satisfy (3.53), can also be interesting and are in fact studied, for neutral blackfolds, in [29, 30].

Since the action (3.28) is stationary under variations of the embedding, with T, Ω_i and Φ_H held constant, we see that solutions to the blackfold equations satisfy the first law (see [13, 31])

$$dM = TdS + \sum_i \Omega_i dJ_i + \Phi_H dQ \tag{3.54}$$

under variations of the embedding functions $X^\mu(\sigma)$. Observe that the scaling dimensions of M, S, J_i and Q are consistent with their numerical coefficients in the Smarr relation.

Let us emphasize that, while the thermodynamic action (3.47), Smarr relation (3.52) and first law (3.54) are exactly valid for all black holes with q-brane charges or dipoles, eqs. (3.49) and (3.50) instead hold only to leading order in the limit where the blackfold construction applies.

Scalings. Let us assume that all length scales along $\mathcal{B}_p$ are of the same order $\sim R$ and that there are no large redshifts, of gravitational or Lorentz type, over most of the blackfold (this is naturally satisfied since the redshifts become large only close to the boundaries). Then we have

$$\begin{aligned} M &\sim R^p r_0^n (1+\hat\nu \sinh^2\alpha)\,, \qquad J \sim R^{p+1} r_0^n\,, \\ Q &\sim R^{p-q} r_0^n \sinh\alpha\cosh\alpha\,, \qquad S \sim R^p r_0^{n+1} (\cosh\alpha)^N\,, \end{aligned} \tag{3.55}$$

where $\hat\nu$ is a non-zero pure number, obtained from n and N, which we shall assume to be of order one. They satisfy

$$S(M,J,\alpha) \sim J^{-\frac{p}{n}} M^{\frac{D-2}{n}} \frac{(\cosh\alpha)^N}{(1+\hat\nu\sinh^2\alpha)^{\frac{D-2}{n}}}\,, \tag{3.56}$$

i.e.,

$$S(M,J,\Phi_H) \sim J^{-\frac{p}{n}} M^{\frac{D-2}{n}} f(\Phi_H)\,, \tag{3.57}$$

so the entropy scales with J and M in the same way as in the neutral case, and is modified by only a factor of a function of the potential.

4 Regime of applicability of blackfolds. Extremal and near-extremal limits

The blackfold approach is applicable whenever a black hole can be approximated locally by a black brane. This requires a separation of scales on the horizon, with one large length along the directions that can be regarded as the worldvolume of the brane, and a small length measuring the size of the horizon in directions transverse to the worldvolume, or more generally the length over which the gravitational field close to the brane in directions transverse to the worldvolume differs appreciably from flatness. The latter scale is the one that gets integrated out in the effective description and allows to treat the brane as a probe in a background spacetime.

The long scale is what we have referred to above as R, while for the small scale, charged blackfolds introduce two different radii: one is the energy-density radius of the black brane,

$$r_\varepsilon \sim r_0(1+\hat{\nu}\sinh^2\alpha)^{1/n} \sim r_0(\cosh\alpha)^{2/n} \tag{4.1}$$

and the other is the charge-density radius

$$r_\mathcal{Q} \sim r_0(\sinh\alpha\cosh\alpha)^{1/n}\,, \tag{4.2}$$

(these are convenient, but other choices, like the horizon radius of the S^{n+1}, could have been made). Since $r_\mathcal{Q}$ cannot be parametrically larger than r_ε, we choose the latter to define the length scale of the region that is integrated out. Then the approach is applicable when $r_\varepsilon \ll R$, *i.e.,* when

$$\frac{r_0}{R} \ll (\cosh\alpha)^{-2/n}\,. \tag{4.3}$$

In the presence of charge densities, r_0 is not the horizon size. Instead, it can be regarded as a 'non-extremality parameter', which can be made very small while physical quantities like mass and charge remain finite and approach the extremal bounds.

Let us now introduce the mass-length and angular momentum-length as in [19],

$$\ell_M \sim (GM)^{\frac{1}{D-3}}\,, \qquad \ell_J \sim \frac{J}{M}\,. \tag{4.4}$$

According to (3.55),

$$\ell_J \sim \frac{R}{1+\hat{\nu}\sinh^2\alpha} \tag{4.5}$$

which implies that, in contrast to neutral blackfolds, the angular momentum-length does not always set the size of the blackfold, R, since ℓ_J can be much smaller than R when extremality is approached. Furthermore,

$$\left(\frac{\ell_M}{\ell_J}\right)^{D-3} \sim \left(\frac{r_0}{R}\right)^n (1+\hat{\nu}\sinh^2\alpha)^{D-2} \sim \left(\frac{r_0}{R}\right)^n (\cosh\alpha)^{2(D-2)}\,, \tag{4.6}$$

which when combined with (4.3) implies that the blackfold approach requires

$$\frac{\ell_M}{\ell_J} \ll \cosh^2\alpha\,. \tag{4.7}$$

When the charge parameter α is moderate, this regime of validity is the ultraspinning regime $\ell_M \ll \ell_J$. However, when α is very large and therefore the blackfold is close to having maximal (extremal) charge, this condition does not impose any hierarchy between ℓ_M and ℓ_J and in particular a large enough α may allow $\ell_J \ll \ell_M$.

Thus there may exist regimes of slowly rotating black holes which can be described as blackfolds, as long as they are sufficiently near extremality. However, the converse statement that the rotation must be small (*i.e.,* $\ell_J \lesssim \ell_M$) for any near-extremal blackfold, need not be true. As we will see now, it is true for $q = 0$ charged blackfolds, but it never occurs for $q = 1$ string-dipole blackfolds.

Using (3.47), (3.49) and (3.52) we derive

$$M - (q+1)\Phi Q = \frac{D-2}{n} TS \,, \tag{4.8}$$

and

$$\sum_i \Omega_i J_i - q\Phi_H Q = \frac{p}{n} TS \,. \tag{4.9}$$

These relations are valid for black holes constructed as blackfolds in backgrounds where $R_0 = 1$, *e.g.,* Minkowski backgrounds. Close to extremality the term TS is very small, so

$$M \simeq (q+1)\Phi_H Q \,, \qquad \sum_i \Omega_i J_i \simeq q\, \Phi_H Q \,. \tag{4.10}$$

Thus we see that for near-extremal charged black holes ($q = 0$) the rotation must be small. Since in this regime $\Phi_H \simeq \sqrt{N}$, the mass approaches the BPS bound $M = \sqrt{N} Q$ for the theories (1.1). This approach to extremality can happen under two different circumstances:

1. The vanishing of ΩJ may signal a breakdown of the blackfold approximation, in which not only the rotation, but indeed the whole black hole horizon becomes small in the perturbative blackfold expansion: R becomes comparable to r_ε so the black hole does not resemble a brane anymore. We expect that this horizon becomes fairly rounded in all its dimensions, *i.e.,* qualitatively similar to the extremal Kerr-Newman solution.

2. The blackfold approximation may instead remain valid, and the worldvolume remain of finite size, while satisfying (4.3), but the brane approaches extremality. In the limit, the fluid becomes a 'charged dust', with no pressure on the worldvolume. Static extremal black holes of Einstein-Maxwell-dilaton theory satisfy a no-force condition that allows to construct exact solutions, in the Majumdar-Papapetrou class, using harmonic functions with arbitrary distributions of sources, in particular continuous ones (all of which have singular horizons). When these are distributed along a p-dimensional submanifold, we obtain a brane of static charged dust. The blackfold approach describes how this extremal singular brane can be 'thermalized' into a near-extremal black brane with a regular horizon, which requires only a small rotation to balance the small tensions that appear. Thus, such black holes have horizons which are much longer in some directions than in others without being ultraspinning.

These two possibilities occur in the solutions that we construct in the next sections.

Let us now analyze blackfolds with $q = 1$ string dipoles. In the extremal limit we find

$$\sum_i \Omega_i J_i = \Phi_H Q = \frac{M}{2}\,. \tag{4.11}$$

In this case the rotation cannot vanish at extremality, since the string energy $\Phi_H Q$ sets a lower limit to the total rotational energy ΩJ. The reason is that, even if the strings may satisfy a no-force condition in directions transverse to them, one still needs to counterbalance their tension with centrifugal repulsion along directions parallel to them.

Eq. (4.11) says that the energy of extremal black holes with string dipole is equally distributed ('virialized') among the total momentum ΩJ and the energy in string tension $\Phi_H Q$. Such relations had already been identified for a variety of constructions (in supergravity and in string theory) of dipole rings with $N = 1$ in [33]. We have now shown that they hold for all extremal black holes with string dipoles, in the ultraspinning limit where the leading order blackfold approximation applies and backreaction effects can be neglected. At the next order, the self-interaction of the black brane modifies its energy and (4.11) receives corrections.

Note also that the relation (4.11) is not a BPS bound. Solutions with only dipoles, and no net charges, cannot be supersymmetric.

4.1 Extremal and near-extremal limits of known charged rotating black holes

The existence of extremal and near-extremal black holes that admit a description as blackfolds can be illustrated using known exact solutions. To this effect we shall consider the charged rotating black holes in Kaluza-Klein theory (*i.e.,* $N = 1$) in $D \geq 6$ as presented in [9], and the five-dimensional charged rotating rings of the same theory in [5, 6, 7]. A few other examples, *e.g.,* with $N = 2$, could be analyzed but they do not add any new features. The extremal limit of solutions with string dipoles can be studied in the dipole black rings of [13], but this analysis has been essentially done already in [33] so we will omit it here.

Let us first study the extremal limit of the KK MP black holes. We refer to appendix C.1 for the solution and the definition of quantities in this section. The extremal limit we are interested in is obtained by taking

$$\alpha \to \infty\,, \qquad m \to 0\,, \qquad m \sinh^2 \alpha = \hat{m} \;\; \text{fixed,} \tag{4.12}$$

while the rotation parameters a_i remain fixed. In this limit the metric becomes

$$ds^2 = -h^{-\frac{D-3}{D-2}} dt^2 + h^{\frac{1}{D-2}} ds^2(\mathbb{R}^{D-1}) \tag{4.13}$$

where

$$ds^2(\mathbb{R}^{D-1}) = F dr^2 + \epsilon r^2 d\nu^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{4.14}$$

is the metric of the flat space $\mathbb{R}^{D-1}$ in spheroidal coordinates, as can be seen by changing to coordinates

$$\rho_i = \sqrt{r^2 + a_i^2}\, \mu_i\,, \qquad z = r\,\nu\,, \tag{4.15}$$

in which

$$ds^2(\mathbb{R}^{D-1}) = \epsilon dz^2 + d\rho_i^2 + \rho_i^2 d\phi_i^2 \,. \tag{4.16}$$

The function

$$h = 1 + \frac{\hat{m} r^{2-\epsilon}}{\Pi F} \tag{4.17}$$

can readily be seen to be a harmonic function in $\mathbb{R}^{D-1}$ (see app. C.1 for the definition of ϵ, Π, F etc.). Since also the electric field and the dilaton become

$$A_t = \left(h^{-1} - 1\right) dt \,, \qquad \phi = -\frac{1}{4} a^{\mathsf{KK}} \ln h \,. \tag{4.18}$$

we see that the solution belongs in the Majumdar-Papapetrou class in D spacetime dimensions.

The conventional static extremal limit is recovered when all $a_i = 0$, so $h = 1 + \hat{m}/r^{D-3}$ is a harmonic function with a pole at $r = 0$ where the solution has a naked pointlike singularity. We consider now a more general case where a number

$$s < \frac{D-3}{2} \,, \tag{4.19}$$

of the possible rotation parameters a_j, $j = 1, \ldots, s$ are non-zero, while the others vanish, $a_k = 0$, $k = s+1, \ldots, \left\lfloor \frac{D-1}{2} \right\rfloor$. The condition (4.19) means that if D is even at least one of the rotation parameters vanishes, and if D is odd at least two of the rotations vanish. Then h has a pole at the locus $r = 0$, but now this is the $2s$-dimensional ball

$$0 \le \rho_j \le a_j \,, \qquad j = 1, \ldots, s \tag{4.20}$$

in (4.16). In this form, we see that the extremal limit is a solution that can be described as a distribution of charged extremal black holes smeared over an even-ball. Since the forces among the extremal black holes cancel, there is no tension on the ball and we can characterize it as a ball of 'charged dust'. Such extremal configurations have singular horizons of zero area (so we incur in a slight misnomer by calling them black holes).

Let us now consider the solution close to the extremal limit, with finite but small m. In this case, the coordinate radius for the horizon, where $\Pi = m r_+^{2-\epsilon}$, can be approximately computed as

$$r_+ \simeq \left(\frac{m}{\prod_{j=1}^{s} a_j^2} \right)^{1/(D-3-2s)} \,. \tag{4.21}$$

The actual dimensions of the horizon are

$$\ell_\perp \simeq r_+ (\cosh\alpha)^{1/(D-2-2s)} \,, \qquad \ell_{\parallel j} \simeq a_j \,, \tag{4.22}$$

in directions orthogonal and parallel to the rotation planes, respectively. We see that the black hole has the pancaked shape, $\ell_\perp \ll \ell_\parallel$, characteristic of the regimes that can be described in the blackfold approach.

However, despite the similarities this is not an ultraspinning regime of the kind studied in [32]: the angular velocities and angular momenta approach zero in the extremal limit,

$$\Omega_i \to \frac{1}{a_i \cosh\alpha} \to 0\,, \qquad J_i \to \frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{m}a_i}{\cosh\alpha} \to 0\,. \tag{4.23}$$

The solution remains of finite extent, but the area and temperature vanish, while the mass and charge remain finite and saturate the BPS bound, as they must for a solution in the class (4.13).

Another example of this limit is provided by the five-dimensional charged black rings of [5, 6, 7]. For simplicity we only consider, again, the charged black ring with $N = 1$, which is given in appendix C.2. The limit we are interested in is

$$\lambda \to 2\nu \to 0\,, \qquad \alpha \to \infty\,, \qquad \lambda \sinh^2\alpha = \hat{q} \ \text{ fixed}, \tag{4.24}$$

while R remains finite, in which the solution becomes

$$ds^2 = -h^{-2/3}dt^2 + h^{1/3}ds^2(\mathbb{R}^4) \tag{4.25}$$

where

$$ds^2(\mathbb{R}^4) = \frac{R^2}{(x-y)^2}\left((y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right) \tag{4.26}$$

is the metric of the flat space $\mathbb{R}^4$ in 'ring' coordinates (see *e.g.,* [34]). So the limiting extremal metric, and also the electric field and the dilaton, are of the same kind as in the previous example, now with a harmonic function

$$h = 1 + \hat{q}(x-y) \tag{4.27}$$

that has poles in a circle of radius R along the direction ψ in $\mathbb{R}^4$. Thus this solution is interpreted as a static ring of charged dust. Near the extremal limit, where $\lambda \simeq 2\nu$ are small and α is large, the radius of the S^2 of the ring is

$$r_+ \simeq R\,\hat{q}^{1/4}\,\nu^{3/4}\,, \tag{4.28}$$

which is much smaller than the S^1 radius R. We have found again that the solution becomes black brane(string)-like even if the rotation is small.

5 Black holes with electric charge

Using the formalism for stationary charged blackfold developed in section 3, now we build compact stationary blackfolds carrying electric charge in a Minkowski background. They describe electrically charged black holes in an asymptotically flat spacetime.

5.1 Disk solution: rotating charged black hole of spherical topology in $D \geq 6$

We start with a simple rotating two-dimensional disk embedded in Minkowski spacetime. The horizon of the corresponding black hole will be topologically a sphere S^{n+1} fibered over the disk $\mathcal{B}_2$. The sphere radius vanishes on the boundary of the disk, resulting in a spherical S^{n+3} horizon.

Since the radius of the disk is much larger than the radius of the internal sphere, the event horizon is not round, but flattened along the blackfold directions. In the neutral case, it was shown in [21] that this configuration reproduces the physics of a pancaked MP black hole in $D = n + 5$ dimensions, with a single ultraspin. Here we add electric charge to this system. For generic dilaton coupling the solutions are new, but in Kaluza-Klein theory (when $N = 1$) we can compare with the known exact solution and find exact agreement. In the non-dilatonic case, the system will describe for the first time the electrically charged generalization of MP black holes in pure Einstein-Maxwell theory in the regimes described in sec. 4.

Let us consider Minkowki spacetime as background, for which $R_0 = 1$. Carter's equations are trivially solved for a flat embedding, so we can restrict the analysis to the blackfold plane, with polar coordinates (r, ϕ). The metric is

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 \,. \tag{5.1}$$

Stationarity implies the fluid is rigidly rotating, and its velocity is

$$u = \gamma \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) , \qquad \gamma = \frac{1}{\sqrt{1 - \Omega^2 r^2}} \,. \tag{5.2}$$

Then, the intrinsic blackfold equations are solved by (3.8) and (3.20), *i.e.*,

$$\mathcal{T} = \gamma T, \qquad \Phi = \gamma \Phi_H. \tag{5.3}$$

T and Φ_H are the temperature and potential at the origin $r = 0$ where the blackfold fluid is at rest. The thickness $r_0(r)$ and the charge parameter $\alpha(r)$ are then obtained by solving (3.21),

$$r_0(r) = \frac{n}{4\pi T} \left(1 - \Omega^2 r^2\right)^{\frac{1-N}{2}} \left(1 - \Omega^2 r^2 - \frac{\Phi_H^2}{N}\right)^{N/2} , \qquad \tanh \alpha(r) = \frac{1}{\sqrt{N}} \frac{\Phi_H}{\sqrt{1 - \Omega^2 r^2}}. \tag{5.4}$$

Observe now that the boundary where the radius r_0 vanishes is at $r = r_{\max}$ given by

$$r_{\max} = \frac{1}{\Omega} \sqrt{1 - \frac{\Phi_H^2}{N}} \,. \tag{5.5}$$

This sets the radius of the disk and, in contrast to uncharged disks, occurs before the velocity u becomes lightlike. Instead $r_{\max}$ is the radius at which the brane becomes locally extremal, $\Phi(r_{\max}) = \sqrt{N}$. Thus the disk radius is reduced by the electric charge. The upper bound on the potential, $\Phi_H = \sqrt{N}$, is saturated for extremal blackfolds, for which $\alpha \to \infty$.

Using (5.4) it is straightforward to compute the charges of the blackfold by performing the integrals (3.34) and (3.36) over the disk. They can be easily expressed in terms of hypergeometric functions, and they factorize into the neutral blackfold result, which carries all the (T, Ω) dependence, times functions that depend only on the potential Φ_H through the dimensionless radius $\Omega r_{\max}$,

$$\begin{aligned} M &= \frac{\Omega_{(n+1)}}{8G\Omega^2} \frac{n+3}{n+2} \left(\frac{n}{4\pi T}\right)^n I_M(\Omega r_{\max}), & J &= \frac{\Omega_{(n+1)}}{4G\Omega^3} \frac{1}{n+2} \left(\frac{n}{4\pi T}\right)^n I_J(\Omega r_{\max}), \\ S &= \frac{\pi \Omega_{(n+1)}}{2G\Omega^2} \frac{1}{n+2} \left(\frac{n}{4\pi T}\right)^{n+1} I_S(\Omega r_{\max}), & Q &= \frac{\Omega_{(n+1)}}{8G\Omega^2} \left(\frac{n}{4\pi T}\right)^n \Phi_H I_Q(\Omega r_{\max}) \,, \end{aligned} \tag{5.6}$$

with

$$I_M(x) = \frac{n+2}{n+3}\,x^{nN}\left[\left(1-x^2\right){}_2F_1\left(1+\frac{n}{2}(N-1),1;1+\frac{nN}{2};x^2\right)\right.$$
$$+\frac{n+1}{nN+2}x^2{}_2F_1\left(1+\frac{n}{2}(N-1),1;2+\frac{nN}{2};x^2\right)$$
$$\left.-\frac{2x^2}{(nN+4)(nN+2)}{}_2F_1\left(1+\frac{n}{2}(N-1),2;3+\frac{nN}{2};x^2\right)\right], \quad (5.7)$$

$$I_J(x) = \frac{n(n+2)}{(nN+4)(nN+2)}x^{nN+4}\,{}_2F_1\left(1+\frac{n}{2}(N-1),2;\frac{nN}{2}+3;x^2\right)$$
$$+\frac{n+2}{nN+2}\left(1-x^2\right)x^{nN+2}\,{}_2F_1\left(1+\frac{n}{2}(N-1),2;\frac{nN}{2}+2;x^2\right), \quad (5.8)$$

$$I_Q(x) = \frac{1}{N}x^{nN}\,{}_2F_1\left(\frac{n}{2}(N-1),1;\frac{nN}{2}+1;x^2\right), \quad (5.9)$$

$$I_S(x) = \frac{n+2}{nN+2}x^{nN+2}\,{}_2F_1\left(\frac{n}{2}(N-1),1;\frac{nN}{2}+2;x^2\right). \quad (5.10)$$

In the uncharged limit we have $\Omega r_{\max} = 1$ and these functions simplify to $I_M(1) = I_J(1) = I_Q(1) = I_S(1) = 1$. In Kaluza-Klein theory, for which $N = 1$, the integrals (5.10) can be expressed in terms of elementary functions. We obtain

$$M = \frac{\Omega_{(n+1)}}{8G(n+2)\Omega^2}\left(\frac{n}{4\pi T}\right)^n\left(1-\Phi_H^2\right)^{\frac{n}{2}}\left(n+3-\Phi_H^2\right), \quad J = \frac{\Omega_{(n+1)}}{4G(n+2)\Omega^3}\left(\frac{n}{4\pi T}\right)^n\left(1-\Phi_H^2\right)^{\frac{n}{2}+1}$$
$$S = \frac{\pi\Omega_{(n+1)}}{2G(n+2)\Omega^2}\left(\frac{n}{4\pi T}\right)^{n+1}\left(1-\Phi_H^2\right)^{\frac{n}{2}+1}, \qquad Q = \frac{\Omega_{(n+1)}}{8G\Omega^2}\left(\frac{n}{4\pi T}\right)^n\Phi_H(1-\Phi_H^2)^{\frac{n}{2}}, \quad (5.11)$$

In appendix C.1 we show that this blackfold result agrees precisely with the exact analytic form of the Kaluza-Klein black hole of [9] in the ultraspinning regime. This is good evidence that the charged disk construction, with its new kind of boundary where the disk velocity remains timelike, results into non-singular horizons. We emphasize that the charges (5.6) describe for the first time $N \neq 1$ electrically charged black holes in the limits where our approximations apply.

In agreement with (3.57), the entropy scales with mass and angular momentum with the same powers as in the neutral case. Finally, the charge per unit mass of these black holes is bounded from above, and obeys the same inequality (A.7) as static charged black holes,

$$\frac{Q}{M} \leq \frac{1}{\sqrt{N}}. \quad (5.12)$$

As discussed in sec. 4, these bounds are saturated for generic extremal charged blackfolds.

Extremal limit. In the extremal limit $\Phi_H \to \sqrt{N}$ we have $\Omega r_{\max} \to 0$ and therefore if Ω remains finite the blackfold disk appears to vanish. More appropriately, the black hole horizon becomes of size $\sim r_Q$ in all its directions, which pushes it outside the applicability of the blackfold approximation. This is an explicit illustration of the first of the two possibilities of extremal limits of charged blackfolds discussed in sec. 4.

The second possibility discussed in sec. 4 is also realized. We could have $\Omega \to 0$ as the extremal limit is approached, and by appropriately tuning the rate at which Ω vanishes, the disk can have arbitrary radius $r_{\max}$. Thus we approach the extremal solution through a family of near-extremal, slowly rotating charged black holes that are flattened on the rotation plane. In the extremal limit we find a disk of charged dust, which can be found as an exact solution of the Einstein-Maxwell-dilaton theory with a singular horizon. The horizon of the near-extremal solutions gets pancaked without being ultraspinning. For Kaluza-Klein dilaton coupling $N = 1$ these are the solutions described in sec. 4.1.

5.2 Even-ball solution: charged black holes with multiple ultraspins

The charged disk solution can be easily generalized to general even-dimensional balls, with ellipsoidal shape. We take $p = 2k$ and work in flat Minkowski background with metric ($i = 1, \ldots, k = p/2$)

$$ds^2 = -dt^2 + \sum_{i=1}^{k} \left(dr_i^2 + r_i^2 d\phi_i^2 \right) . \tag{5.13}$$

The velocity of the rigidly rotating fluid describing the stationary blackfold is given by

$$u = \gamma \left(\frac{\partial}{\partial t} + \sum_{i=1}^{k} \Omega_i \frac{\partial}{\partial \phi_i} \right) , \qquad \gamma = \frac{1}{\sqrt{1 - \sum_{i=1}^{k} \Omega_i^2 r_i^2}} . \tag{5.14}$$

Then, the intrinsic blackfold equations are again solved by the correctly redshifted temperature T and potential Φ_H (5.3) as for the disk solution. The thickness $r_0(r)$ and the charge parameter $\alpha(r)$ are

$$r_0(r_i) = \frac{n}{4\pi T} \left(1 - \sum \Omega_i^2 r_i^2 \right)^{\frac{1-N}{2}} \left(1 - \sum \Omega_i^2 r_i^2 - \frac{\Phi_H^2}{N} \right)^{\frac{N}{2}} , \quad \tanh \alpha(r_i) = \frac{\Phi_H / \sqrt{N}}{\sqrt{1 - \sum \Omega_i^2 r_i^2}} . \tag{5.15}$$

The thickness of the brane vanishes on the boundary of the ellipsoid $\mathcal{E}$ defined by

$$\sum_{i=1}^{k} \Omega_i^2 r_i^2 \le x_m^2 , \qquad x_m = \sqrt{1 - \frac{\Phi_H^2}{N}} , \tag{5.16}$$

From this we can also see that we have to require that $\Phi_H^2 \le N$. We can now calculate the charges by performing the integrals (3.34)-(3.36) on $\mathcal{E}$. The dependences on T, Ω_i and Φ_H factorize and

we find that the charges are given by the uncharged results times a potential dependent term,

$$M = \frac{(2\pi)^{\frac{p}{2}}\Omega_{(n+1)}}{16\pi G\prod\Omega_i^2}\left(\frac{n}{4\pi T}\right)^n I_M(x_m), \qquad J_j = \frac{(2\pi)^{\frac{p}{2}}n\Omega_{(n+1)}}{16\pi G\Omega_j\prod\Omega_i^2}\left(\frac{n}{4\pi T}\right)^n I_{J_j}(x_m), \tag{5.17}$$
$$S = \frac{(2\pi)^{\frac{p}{2}}\Omega_{(n+1)}}{4G\prod\Omega_i^2}\left(\frac{n}{4\pi T}\right)^{n+1} I_S(x_m), \qquad Q = \frac{n(2\pi)^{\frac{p}{2}}\Omega_{(n+1)}}{16\pi G\prod\Omega_i^2}\left(\frac{n}{4\pi T}\right)^n \Phi_h I_Q(x_m),$$

where the electric potential enters through the functions

$$I_M(x_m) = \int_{\mathbf{x}^2\leq x_m^2}\left(\prod_i x_i dx_i\right)\left(1-\mathbf{x}^2\right)^{\frac{n}{2}(1-N)-1}\left(x_m^2-\mathbf{x}^2\right)^{nN/2}\left[n+1-\mathbf{x}^2+\frac{nN\left(1-x_m^2\right)}{x_m^2-\mathbf{x}^2}\right],$$
$$I_{J_j}(x_m) = \int_{\mathbf{x}^2\leq x_m^2}\left(\prod_i x_i dx_i\right)x_j^2\left(1-\mathbf{x}^2\right)^{\frac{n}{2}(1-N)-1}\left(x_m^2-\mathbf{x}^2\right)^{nN/2}\left[1+N\frac{1-x_m^2}{x_m^2-\mathbf{x}^2}\right],$$
$$I_Q(x_m) = \int_{\mathbf{x}^2\leq x_m^2}\left(\prod_i x_i dx_i\right)\left(1-\mathbf{x}^2\right)^{\frac{n}{2}(1-N)}\left(x_m^2-\mathbf{x}^2\right)^{nN/2-1},$$
$$I_S(x_m) = \int_{\mathbf{x}^2\leq x_m^2}\left(\prod_i x_i dx_i\right)\left(1-\mathbf{x}^2\right)^{\frac{n}{2}(1-N)}\left(x_m^2-\mathbf{x}^2\right)^{nN/2}. \tag{5.18}$$

We used $\mathbf{x}^2$ as a shorthand for $\sum x_i^2$. With $p=2$, these expression reproduce the results for the rotating disk of section 5.1, and in the uncharged ($\Phi_H = 0$) limit we recover the neutral even-ball quantities of [21].

5.3 Odd-sphere solution with electric charge

A natural solution to seek is the thin charged black ring. As shown in [18, 21], the blackfold construction of the ring solution is the first occurrence of a larger family of blackfold solutions whose worldvolumes wrap an odd-dimensional sphere S^p, with $p = 2k+1$. In general, with non-equal rotations Ω_i on the Cartan planes of the sphere, one has to solve a complicated differential equation for the blackfold embedding. However, for equal rotations Ω on all planes, the S^p is a round sphere of radius R and the resulting solution is simple. We will therefore restrict to this simple case in the analysis of odd-sphere blackfolds, and regard the black ring as a particular case that can be recovered for $p=1$ as there is no added complexity in considering a general p.

We embed the S^p sphere in $\mathbb{R}^{p+1}$ with metric

$$dr^2 + r^2\sum_{i=1}^{k+1}\left(d\mu_i^2+\mu_i^2 d\phi_i^2\right), \qquad \sum_{i=1}^{k+1}\mu_i^2 = 1, \tag{5.19}$$

as a constant $r=R$ hypersurface. Then, the sphere is parameterized by $k+1$ Cartan angles ϕ_i and k independent director cosines μ_i. The velocity of a fluid rigidly rotating with equal angular velocity Ω in all $k+1$ Cartan planes is given by

$$u = \gamma\left(\frac{\partial}{\partial t}+\Omega\sum_{i=1}^{k+1}\frac{\partial}{\partial\phi_i}\right), \qquad \gamma = \frac{1}{\sqrt{1-\Omega^2R^2}}. \tag{5.20}$$

The extrinsic curvature of the sphere is easily seen to be $K^r = -p/R$ and then the extrinsic equation (2.43) is solved by choosing its radius to satisfy

$$R = \frac{1}{\Omega}\sqrt{\frac{p}{n+p+nN\sinh^2\alpha}}. \tag{5.21}$$

This is the equilibrium radius for which the centrifugal force due to the rotation of the blackfold balances the tension of the sphere. In terms of the rapidity η, with $\tanh\eta = \Omega R$, (5.21) is

$$\sinh^2\eta = \frac{p}{n(1+N\sinh^2\alpha)}, \tag{5.22}$$

and we see that the local velocity needed to support the blackfold is reduced by the presence of charge, as anticipated in sec. 2.3. The fluid velocity decreases from the neutral limit $\Omega R = \sqrt{p/(n+p)}$ to a configuration with $\Omega R = 0$ in the extremal limit $\alpha \to \infty$, $\Phi_H \to \sqrt{N}$. We will return to this limit at the end of our analysis of the solutions.

The intrinsic equations are solved by (5.3). The temperature T and the potential Φ_H are constant over the sphere. Using (5.21), the latter can be expressed as

$$\Phi_H = \sqrt{nN}\sqrt{\frac{1+N\sinh^2\alpha}{n+p+nN\sinh^2\alpha}}\tanh\alpha. \tag{5.23}$$

To find the explicit dependence of the sphere radius $R(\Omega,\Phi_H)$ on the angular velocity and electric potential, one must solve equation (5.23) for α and substitute the result in (5.21) leading to

$$R = \frac{\sqrt{n+2p+(nN-n-p)\frac{\Phi_H^2}{N} - \sqrt{\left(n-(nN+n+p)\frac{\Phi_H^2}{N}\right)^2 + 4n(n+p)\Phi_H^2\left(1-\frac{\Phi_H^2}{N}\right)}}}{\Omega\sqrt{2(n+p)}}. \tag{5.24}$$

For simplicity, we choose to express all quantities in terms of α instead of Φ_H. Then, the constant thickness of the blackfold in terms of T is

$$r_0 = \frac{n^{3/2}}{4\pi T}\left(\frac{1+N\sinh^2\alpha}{n+p+nN\sinh^2\alpha}\right)^{1/2}(\cosh\alpha)^{-N}\ . \tag{5.25}$$

Since the worldvolume fields are constant, it is straightforward to compute the thermodynamic variables for these blackfolds. The integrals over the blackfold worldvolume reduce to a simple multiplication by the size $R^p\Omega_{(p)}$ of the sphere S^p. Then

$$M = \frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{p/2}}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{\left(1+N\sinh^2\alpha\right)^{n/2}\left(n+p+1+nN\sinh^2\alpha\right)}{\left(n+p+nN\sinh^2\alpha\right)^{(n+p)/2}(\cosh\alpha)^{nN}}, \tag{5.26}$$

$$J = \frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{1+p/2}}{\Omega^{p+1}}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{\left(1+N\sinh^2\alpha\right)^{n/2}}{\left(n+p+nN\sinh^2\alpha\right)^{\frac{n+p}{2}}(\cosh\alpha)^{nN}}, \tag{5.27}$$

$$S = \frac{\Omega_{(n+1)}\Omega_{(p)}}{4G}\frac{p^{p/2}}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^{n+1} \frac{\left(1+N\sinh^2\alpha\right)^{n/2}}{n^{1/2}\left(n+p+nN\sinh^2\alpha\right)^{(n+p)/2}(\cosh\alpha)^{nN}}, \tag{5.28}$$

$$Q = \frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{p/2}}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{\sqrt{nN}\sinh\alpha\left(1+N\sinh^2\alpha\right)^{(n-1)/2}}{\left(n+p+nN\sinh^2\alpha\right)^{(n+p-1)/2}(\cosh\alpha)^{nN-1}}, \tag{5.29}$$

Here J is the momentum conjugate to $\sum_{i=1}^{k+1}\partial_{\phi_i}$, and $\alpha=\alpha(\Phi_H)$ is given implicitly by (5.23).

In can be checked that the entropy of these black holes scales with the mass and angular momentum exactly like their neutral counterparts, as predicted by (3.57). Moreover, their charge to mass ratio satisfies precisely the same bound (5.12) as the disk solutions and static charged black holes.

As an example in which the formulas do not get too cumbersome, we can solve in the KK case ($N=1$) the relation between the parameter α and the potential Φ_H, to obtain

$$\tanh\alpha=\Phi_H\sqrt{\frac{n+p}{n+p\Phi_H^2}}\,. \tag{5.30}$$

In this case, the electric potential is bounded according to $\Phi_H\leq 1$ and the charges can be written explicitly in terms of the potentials,

$$M=\frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{p/2}}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^n\left(\frac{1-\Phi_H^2}{n+p}\right)^{\frac{n+p}{2}}\frac{n+p+1-\Phi_H^2}{1-\Phi_H^2}, \tag{5.31}$$

$$J=\frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{\frac{p}{2}+1}}{\Omega^{p+1}}\left(\frac{n^{3/2}}{4\pi T}\right)^n\left(\frac{1-\Phi_H^2}{n+p}\right)^{\frac{n+p}{2}}, \tag{5.32}$$

$$S=\frac{\Omega_{(n+1)}\Omega_{(p)}}{4G}\frac{p^{p/2}}{\sqrt{n}\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^{n+1}\left(\frac{1-\Phi_H^2}{n+p}\right)^{\frac{n+p}{2}}, \tag{5.33}$$

$$Q=\frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{p^{p/2}}{\Omega^p}\Phi_H\left(\frac{n^{3/2}}{4\pi T}\right)^n\left(\frac{1-\Phi_H^2}{n+p}\right)^{\frac{n+p}{2}-1}. \tag{5.34}$$

We stress that all these results describe electrically charged black rings in the ultraspinning or near-extremal regimes when $p=1$. In the $N=1$ case, an exact solution of this electric black ring exists, and agrees with our results in the ultraspinning limit. We show this in appendix C.1. The near-extremal case was considered in sec. 4.

Extremal limit. We have found above that $\Omega R\to 0$ in the extremal limit of these solutions. It is easy to see that also $\Omega J\to 0$, which is in agreement with our general result in sec. 4 for $q=0$. The two possible realizations of this limit that we discussed there, also occur in this example. The first one results if we keep Ω finite, so R must vanish. This really means that the size of the blackfold sphere S^p becomes too small, on the scale of r_Q, and thus beyond the leading order of the blackfold perturbative construction. In this sense, this limit is similar to what we have found for the extremal limit of disks keeping Ω finite, only now with several equal spins instead of just one.

The second interpretation obtains if we keep R finite, so now the solution is not small, but is static. It is in fact a sphere of charged dust, and the exact solution of Einstein-Maxwell-dilaton theory that it corresponds to can be obtained in the Majumdar-Papapetrou form with harmonic functions sourced on S^p, *i.e.,* it is a spherical smeared distribution of extremal static black holes. Its

horizon is singular, but it can be regarded as a 'good' singularity since by making it non-extremal, which requires a small rotation, it becomes regular.

5.4 General products of odd-spheres

In [21] it was shown that the blackfold worldvolume can wrap any product of odd-spheres, obtaining black holes in D-dimensional flat space with horizon topology

$$\left(\prod_{p_a=\text{odd}} S^{p_a}\right) \times S^{n+1}, \qquad \sum_{a=1}^{l} p_a = p. \tag{5.35}$$

We now obtain their charged generalization. Following the notation of [21], we embed the blackfold in Minkowski spacetime with metric

$$dr_a^2 + r_a^2 \sum_{a,i} \left(d\mu_{a,i}^2 + \mu_{a,i}^2 d\phi_{a,i}^2\right), \qquad \sum_{i=1}^{k_a+1} \mu_{a,i}^2 = 1, \tag{5.36}$$

where the ath sphere is parameterized by the director cosines and Cartan angles $(\mu_{a,i}, \phi_{a,i})$, in analogy to (5.19). Then, in a gauge where the spatial worldvolume coordinates are given by the $(\mu_{a,i}, \phi_{a,i})$, the embedding of the blackfold worldvolume $\mathcal{B}_p$ is described by a collection of scalars $r_a = R_a(\mu_1, \ldots, \mu_k)$ that depend only on the director cosines, since we want to preserve the rotational symmetry of the Cartan planes. The induced worldvolume metric of the blackfold is then given by

$$ds_p^2 = \sum_{a=1}^{l} \left[\sum_{i,j=1}^{k_a} \left[\left(\delta_{ij} + \frac{\mu_{a,i}\mu_{a,j}}{\mu_{a,k_a+1}^2} \right) R_a^2 + \partial_{a,i} R_a \partial_{a,j} R_a \right] d\mu_{a,i} d\mu_{a,j} + R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2 d\phi_{a,i}^2 \right]. \tag{5.37}$$

The velocity field of a fluid rigidly rotating along the angles $\phi_{a,i}$ is,

$$u = \gamma \left(\frac{\partial}{\partial t} + \sum_{a,i} \Omega_{a,i} \frac{\partial}{\partial \phi_{a,i}} \right), \qquad \gamma = \frac{1}{\sqrt{1 - \sum_a R_a^2 \sum_i \mu_{a,i}^2 \Omega_{a,i}^2}}, \tag{5.38}$$

and one can determine the set of equations that the embedding functions $R_a(\mu_{a,i})$ must satisfy by writing down the worldvolume action (3.31) and extremizing it. This yields a complicated set of equations; the example for a neutral S^3 blackfold can be found in [21] and we shall not attempt to do it here. In the simpler case in which the scalars R_a are constant defining the radii of the l spheres wrapped by the blackfold, and the angular momenta of the ath sphere are all equal $(\Omega_{a,i} = \Omega_i)$, the equations become algebraic and can be easily solved to find the equilibrium conditions

$$\Omega_a R_a = \sqrt{\frac{p_a}{n + p + nN \sinh^2 \alpha}} = \sqrt{\frac{p_a}{p}}\, \Omega R\,, \tag{5.39}$$

where ΩR is the result, (5.21) or (5.24), for the equilibrium of a single odd p-sphere blackfold.

With the shape of the blackfold at hand, the intrinsic equations are solved as usual by (3.21), and the thickness of the transverse S^{n+1} is

$$r_0(\mu_{a,i}) = \frac{n}{4\pi T}\left(1 - \sum_a R_a^2 \sum_i \mu_{a,i}^2 \Omega_{a,i}^2\right)^{\frac{1-N}{2}} \left(1 - \sum_a R_a^2 \sum_i \mu_{a,i}^2 \Omega_{a,i}^2 - \frac{\Phi_H^2}{N}\right)^{N/2} . \tag{5.40}$$

The charges are finally easily found to be

$$M = \frac{(2\pi)^{\frac{p+l}{2}}\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n I_M(x_m,\Omega), \tag{5.41}$$

$$J_{b,j} = \frac{n(2\pi)^{\frac{p+l}{2}}\Omega_{(n+1)}\Omega_{b,j}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n I_{J_{b,j}}(x_m,\Omega) \tag{5.42}$$

$$Q = \frac{n(2\pi)^{\frac{p+l}{2}}\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \Phi_h I_Q(x_m,\Omega), \tag{5.43}$$

$$S = \frac{(2\pi)^{\frac{p+l}{2}+1}\Omega_{(n+1)}}{8\pi G}\left(\frac{n}{4\pi T}\right)^{n+1} I_S(x_m,\Omega) \tag{5.44}$$

where the integrals are functions of the angular velocities, Ω_i, and the electric potential, given by

$$\begin{aligned}
I_M(x_m,\Omega) &= \int_{\tilde{\mathcal{B}}} d\tilde{B}\left(1 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{n}{2}(1-N)-1}\left(x_m^2 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{nN}{2}} \\
&\quad\times\left(n+1-\sum_a R_a^2\sum_{i=1}^{k_a+1}\mu_{a,i}^2\Omega_{a,i}^2 + \frac{nN\left(1-x_m^2\right)}{x_m^2 - \sum_a R_a^2\sum_{i=1}^{k_a+1}\mu_{a,i}^2\Omega_{a,i}^2}\right), \\
I_{J_{b,j}}(x_m,\Omega) &= \int_{\tilde{\mathcal{B}}} d\tilde{B} R_b^2\mu_{b,j}^2\left(1 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{n}{2}(1-N)-1}\left(x_m^2 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{nN}{2}} \\
&\quad\times\left(1 + \frac{N\left(1-x_m^2\right)}{x_m^2 - \sum_a R_a^2\sum_{i=1}^{k_a+1}\mu_{a,i}^2\Omega_{a,i}^2}\right), \\
I_Q(x_m,\Omega) &= \int_{\tilde{\mathcal{B}}} d\tilde{B}\left(1 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{n}{2}(1-N)}\left(x_m^2 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{nN}{2}-1}, \\
I_S(x_m,\Omega) &= \int_{\tilde{\mathcal{B}}} d\tilde{B}\left(1 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{n}{2}(1-N)}\left(x_m^2 - \sum_a R_a^2 \sum_{i=1}^{k_a+1} \mu_{a,i}^2\Omega_{a,i}^2\right)^{\frac{nN}{2}},
\end{aligned} \tag{5.45}$$

with the integration measure defined as

$$d\tilde{B} = \prod_a R_a^{2k_a+1}\left(1 + \mu_{a,k+1}^2\sum_{i=1}^{k_a}(\partial_i \ln R_a)^2\right)^{1/2}\prod_{i=1}^{k_a}(\mu_{a,i}d\mu_{a,i}). \tag{5.46}$$

Note that in all of these, whenever it appears, $\mu_{a,k+1}^2 = 1 - \sum_{i=1}^k \mu_{a,i}^2$. It is clear that this reduces to the case of a single S^{2k+1} sphere if we put $l = 1$.

We conclude this section with a reminder that, as discussed in the introduction, our approach does not capture the correct physics of charged rotating black rings in the presence of Chern-Simons terms, such as the solutions of minimal five-dimensional supergravity ($N = 3$) in [4, 7]. The appearance of Dirac-Misner strings in the rotation form of the S^2 [6, 7] has the consequence that the infinite-radius limit of the ring, as long as it is not a static string of charged dust (as in sec. (4.1)) but retains the momentum, cannot have only 0-brane charge but also carries a string charge, so it cannot be in the class of solutions we have studied. This appears to be an effect entirely due to the Chern-Simons term, which is absent in the five-dimensional theories with $N = 1$ and $N = 2$ for which our construction of thin rings does work correctly. Our construction above should also capture correctly the charged black rings of the Einstein-Maxwell theory without a dilaton ($N = 3$) and *without* Chern-Simons term. Moreover, the possibility of Dirac-Misner strings does not affect black holes with spherical horizon topologies for which the S^{n+1} is non-trivially fibered over the horizon.

6 Black holes with string dipole

Now we apply our formalism to construct blackfolds with strings dissolved in their compact worldvolume. They correspond to black holes with string dipoles.

6.1 Annulus solution: prolate black ring with string dipole in $D \geq 6$

As in sec. 5.1, we begin by looking for black 2-folds with worldvolume spread in axially symmetric way in a geometry (5.1), so the velocity is again of the form (5.2) and the temperature determined according to (3.8). The unit vector v along the string directions is

$$v = \frac{\gamma}{r}\left(\frac{\partial}{\partial\phi} + \Omega r^2 \frac{\partial}{\partial t}\right) \tag{6.1}$$

and the worldsheet area element

$$|\hat{h}|^{1/2} = r\,. \tag{6.2}$$

The potential is then

$$\Phi = \frac{\Phi_H}{2\pi r}\,. \tag{6.3}$$

Solving for $r_0(r)$ and $\alpha(r)$ we get

$$r_0(r) = \frac{n}{4\pi T}\sqrt{1-\Omega^2 r^2}\,\left(1 - \frac{\Phi_H^2}{N(2\pi r)^2}\right)^{N/2}, \tag{6.4}$$

$$\tanh\alpha(r) = \frac{\Phi_H}{\sqrt{N}2\pi r}\,. \tag{6.5}$$

This fixes the solution. When computing the physical quantities we take $n = \partial/\partial t$ and $m = r^{-1}\partial/\partial\phi$.

The worldvolume extends along values of r between zeroes of r_0

$$r_{min} = \frac{\Phi_H}{2\pi\sqrt{N}} \leq r \leq r_{max} = \frac{1}{\Omega}\,. \tag{6.6}$$

Note that both boundary conditions (2.46) and (2.48) are satisfied by this solution. The upper bound, at which u becomes lightlike, is already familiar from disk blackfolds and we expect the horizon to close smoothly there. However, the lower bound is of a new type. Its origin can be understood in physical terms from the Euler equation for the fluid,

$$\partial_r P_\perp = \gamma^2 \left(r\omega^2 \left(\epsilon + P_\perp\right) - \frac{\mathcal{Q}\Phi}{r} \right)\,. \tag{6.7}$$

The first term on the right is the usual centrifugal force. The second term, proportional to $\mathcal{Q}\Phi$, has instead a centripetal effect on the fluid that grows at smaller radii like $1/r$. This is the force due to the strings, which is equal to their tension times the extrinsic curvature, *i.e.,* $\mathcal{Q}\Phi/r$. Since the centrifugal force decreases with r, and the stationary fluid must rotate rigidly, it follows that at sufficiently small r the tension becomes too strong for the fluid to be able to remain in equilibrium. This sets a minimum radius for the fluid, so the worldvolume is not a disk but an annulus.

Since the size of the black brane's S^{n+1} (not only r_0, but also their area) vanishes at the inner and outer edges of the annulus, the horizon of the black hole has the topology of a ring $S^1 \times S^{n+2}$, where the S^{n+2} appears as the fibration of a sphere S^{n+1} of radius r_0 over an interval equal to the radial span $r_{max} - r_{min}$ of the annulus. Since the blackfold approximation requires $r_{max} - r_{min} \gg r_0$ this horizon differs geometrically from that of other black rings in that the S^{n+2} is a prolate sphere, highly elongated in the direction transverse to the strings. In contrast to the flattening caused by centrifugal forces, this elongation of the horizon can be regarded as caused by the repulsion among parallel strings. Therefore this is a qualitatively new class of dipole black rings, which exist only in $D \geq 6$. The connection between the two classes will be clarified below.

As discussed in sec. 3.2, at the boundary at the inner rim where the bound on the potential is reached, the black brane becomes extremal. Nevertheless it remains non-extremal away from that edge.

While we expect from previous examples that the horizon remains smooth at the outer edge, it is less obvious whether it will also remain smooth at the inner edge. We found evidence that the 'extremal edge' was not a problem for charged disks since they correctly reproduced solutions that we know are smooth. In the present case there is no exact solution that we can compare to, but still we will find evidence of good behavior. At the very least, even if the solutions could have a singularity at the inner edge, this would probably be a 'good singularity', as they appear to be connected to other ring solutions with good singularities. Nevertheless, previous experience suggests that the edge in the full solution is likely to be smooth.

The conserved charges are obtained by integrating the densities over the blackfold annulus, and

read

$$
\begin{aligned}
M &= \frac{\Omega_{(n+1)}}{8G\Omega^2}\left(\frac{n}{4\pi T}\right)^n I_M\left(x_m\right), \qquad J = \frac{n\Omega_{(n+1)}}{8G\Omega^3}\left(\frac{n}{4\pi T}\right)^n I_J\left(x_m\right), \\
S &= \frac{\pi\Omega_{(n+1)}}{2G\Omega^2}\left(\frac{n}{4\pi T}\right)^{n+1} I_S\left(x_m\right), \qquad Q = \frac{\Omega_{(n+1)}}{32\pi^2 G} n\Phi_H \left(\frac{n}{4\pi T}\right)^n I_Q\left(x_m\right),
\end{aligned} \tag{6.8}
$$

where $x_m = \Omega r_{min}$ is the dimensionless inner radius and the integrals are given by

$$
\begin{aligned}
I_M(x_m) &= \int_{x_m}^1 \frac{(1-x^2)^{n/2}}{x^{nN-1}}\left(x^2-x_m^2\right)^{nN/2}\left(1+\frac{n}{1-x^2}+\frac{nNx_m^2}{x^2-x_m^2}\right)dx\,, \\
I_J(x_m) &= \int_{x_m}^1 \frac{(1-x^2)^{n/2-1}}{x^{nN-3}}\left(x^2-x_m^2\right)^{nN/2} dx\,, \\
I_Q(x_m) &= \int_{x_m}^1 \frac{(1-x^2)^{n/2}}{x^{nN-1}}\left(x^2-x_m^2\right)^{nN/2-1} dx\,, \\
I_S(x_m) &= \int_{x_m}^1 \frac{(1-x^2)^{n/2}}{x^{nN-1}}\left(x^2-x_m^2\right)^{nN/2} dx\,.
\end{aligned} \tag{6.9}
$$

These integrals converge at the boundaries $x = 1$, $x = x_m$. They can be expressed in terms of hypergeometric functions, but simplify to elementary functions in the non-dilatonic, $D = 6$ case, for which $N = 2$, $n = 1$,

$$
M = \frac{1}{6G\Omega^2 T}\left(1-x_m^2\right)^{3/2}, \quad J = \frac{1}{12G\Omega^3 T}\left(1-x_m^2\right)^{3/2}, \tag{6.10}
$$

$$
Q = \frac{\Phi_H}{32\pi^2 GT}\left(\ln\frac{1+\sqrt{1-x_m^2}}{x_m} - \sqrt{1-x_m^2}\right), \tag{6.11}
$$

$$
S = \frac{1}{8G\Omega^2 T^2}\left[\frac{1}{3}\left(1-x_m^2\right)^{3/2} - x_m^2\left(\ln\frac{1+\sqrt{1-x_m^2}}{x_m} - \sqrt{1-x_m^2}\right)\right]. \tag{6.12}
$$

Here $x_m = \Phi_H \Omega/(\sqrt{8}\pi)$.

Exactly like for the disk solution carrying 0-brane charge, the scaling of the entropy with mass and angular momentum is unchanged with respect to the neutral case and is given by (3.57). However, the Ω-dependence of the charge has changed with respect to the corresponding dependence for 0-brane charged disks, because we integrate the density only on directions transverse to the strings carrying the charge. As a consequence, $Q/M \propto \Omega$, and there is no bound on the charge.

Extremal limit. The annulus solutions exist only as long as

$$
\Phi_H \leq \sqrt{N}\frac{2\pi}{\Omega}\,. \tag{6.13}
$$

When the upper bound here is saturated we have $r_{min} \to r_{max}$, so the annulus becomes infinitely thin and the blackfold becomes both extremal and lightlike. This suggests that this limit corresponds to an extremal black ring with lightlike rotation and with zero temperature. In general all

the functions in (6.8) smoothly go to zero as $\Phi_H \to 2\pi\sqrt{N}/\Omega$ if we keep T finite, but if instead we take the limit as

$$T \to 0\,, \quad \Phi_H \to \sqrt{N}\frac{2\pi}{\Omega}\,, \quad \text{with} \;\; \frac{(2\pi\sqrt{N} - \Phi_H\Omega)^{(1+N)/2}}{T} \;\; \text{finite} \tag{6.14}$$

we can easily extract the dominant limiting terms in (6.9). Taking into account that the ring radius is $R = 1/\Omega$ we find that

$$\sqrt{N}Q2\pi R = \frac{J}{R} = \frac{M}{2}\,. \tag{6.15}$$

These are the correct extremal limit relations derived in (4.11). We regard this as evidence that the annulus blackfolds are physically sensible solutions.

Note, however, that we should not expect that the entropy of the annulus blackfold matches that of a dipole ring in this limit. The annulus is locally equivalent to a (boosted) 2-brane with string charge, and the latter is 'smeared' along the direction transverse to the strings. Thus the annulus can be regarded as a smeared distribution of concentric rings. It is well-known that the extremal horizons do not behave smoothly under smearing — *e.g.,* if an extremal black string with regular horizon is smeared in a transverse direction, the horizon of the resulting string-charged 2-brane is singular. Indeed, the mismatch in the entropies is even stronger in this case, since we are approaching the extremal ring limit not from an extremal 2-brane annulus, but from an annulus that is extremal only at its inner edge. This makes the limiting entropy diverge if $N > 1$. What we are seeing here is that the approximations involved in this blackfold construction break down when $r_Q \sim r_{max} - r_{min}$. Nevertheless, the mass, angular momentum and charge are measured at large distance from the brane and are not sensitive to these horizon effects, hence the good behavior shown in (6.15).

6.2 Solid ring and hollow ball solution: prolate black odd-spheres

Given $p = 2k$, a direct generalization of the annulus solution is given by black $2k$-folds with worldvolume spread in axially symmetric way in the geometry (5.13). As in the electric even-ball case, the velocity field of the fluid takes the form (5.14) and the temperature is fixed by (3.8). To keep the worldvolume compact, all angular velocities Ω_i have to be non vanishing. Smearing the charge-carrying strings on the planes of rotation, the unit vector v along the string directions is determined by the Killing vector

$$\psi = \sum_{i=1}^{k} \varsigma_i \frac{\partial}{\partial \phi_i}, \tag{6.16}$$

through equations (3.13) and (3.15). The constants ς_i determine the orientation, or polarization, of the strings in the worldvolume. Then, according to (3.19), the worldsheet area element is

$$|\hat{h}|^{1/2} = \sqrt{\gamma^{-2}\sum_{i=1}^{k}\varsigma_i^2 r_i^2 + \left(\sum_{i=1}^{k}\varsigma_i\Omega_i r_i^2\right)^2}\,. \tag{6.17}$$

The thickness $r_0(r_i)$ of the brane is given by (3.21), and the worldvolume of the brane fills the region $r_0(r_i) \geq 0$, or equivalently the region defined by the conditions

$$\sum_{i=1}^{k} \Omega_i^2 r_i^2 \leq 1 \qquad \text{and} \qquad \left(1 - \sum_{i=1}^{k} \Omega_i^2 r_i^2\right) \sum_{i=1}^{k} \varsigma_i^2 r_i^2 + \left(\sum_{i=1}^{k} \varsigma_i \Omega_i r_i^2\right)^2 \geq \frac{\Phi_H^2}{(2\pi)^2 N}. \tag{6.18}$$

The first condition simply confines the blackfold worldvolume to an ellipsoid with length of the axes set by the angular velocities Ω_i. On the boundary of this ellipsoid, the velocity u becomes lightlike. The second constraint affects the blackfold shape only in presence of string charges, drilling holes in the planes on which the polarization of these strings lies. This leads to a variety of complicated geometries. We content ourselves to illustrate it in the simple case of $p = 4$ blackfold, by studying two quintessential cases.

First, we align the strings to the rotation, by taking for example $\Omega_1 = \Omega_2 = \Omega$ and $\varsigma_1 = \varsigma_2 = 1$. The blackfold fills now a hollow ball, defined by

$$\frac{\Phi_H}{2\pi\sqrt{N}} \leq r \leq \frac{1}{\Omega}, \tag{6.19}$$

where $r = \sqrt{r_1^2 + r_2^2}$. This is a direct generalization of the annulus blackfold obtained in the previous section. The black hole event horizon has a S^{n+1} fibered over the hollow ball $S^3 \times I$, whose radius vanishes on both exterior and interior boundaries (the endpoints of the interval I), resulting in a topology $S^3 \times S^{n+2}$. This is an odd-sphere solution of a qualitatively different kind than we shall find in the next section: although they share the same topology, the S^{n+2} is strongly elongated in the radial direction of the hollow ball, and assumes a cigar-like shape, with a longitudinal length (of I) much larger than the size of the S^{n+1},

$$\ell_\parallel \sim \frac{1}{\Omega} - \frac{\Phi_H}{2\pi\sqrt{N}} \gg \ell_\perp \sim r_0. \tag{6.20}$$

This construction extends straightforwardly to any even p, leading to prolate odd-sphere blackfolds that describe black holes with $S^{2k-1} \times S^{n+2}$ horizon topology, the S^{n+2} sphere being prolate in the same sense as described above. By varying the parameters (Ω_i, ς_i) one can deform the blackfold sphere S^{2k-1} into a spheroid.

The second representative solution that we exhibit has $\Omega_{1,2} \neq 0$ (and not necessarily equal) and the polarization vector of the string dipole lying in the (r_1, ϕ_1) plane (*i.e.,* $\varsigma_1 = 1, \varsigma_2 = 0$) so the strings and the rotation are not fully aligned, and the fluid is restricted to the region

$$\Omega_1^2 r_1^2 + \Omega_2^2 r_2^2 \leq 1, \qquad r_1^2 \left(1 - \Omega_2^2 r_2^2\right) \geq \frac{\Phi_H^2}{(2\pi)^2 N}. \tag{6.21}$$

The quadrant $r_1, r_2 \geq 0$ represents the quotient space $\mathbb{R}^4/U(1)^2$, where the two $U(1)$'s are the two rotations generated by $\partial_{\phi_{1,2}}$. In this quadrant, as long as

$$\Phi_H \leq \sqrt{N}\frac{2\pi}{\Omega_1}, \tag{6.22}$$

the region covered by (6.21) is non-vanishing, and the two bounding curves can be regarded as describing a lens (*i.e.,* the convex region bounded by two arcs), cut in half by the axis $r_2 = 0$ where

$$\frac{\Phi_H}{2\pi\sqrt{N}} \leq r_1 \leq \frac{1}{\Omega_1}. \tag{6.23}$$

On the other hand r_2 varies between

$$0 \leq r_2 \leq \frac{1}{\Omega_2}\sqrt{1 - \frac{\Omega_1 \Phi_H}{2\pi\sqrt{N}}}, \tag{6.24}$$

with the maximum being reached at $r_1 = \sqrt{\Phi_H/(2\pi\sqrt{N}\Omega_1)}$. The action of ∂_{ϕ_2} on this half-lens turns it into a topological B_3 of lenticular shape. The action of ∂_{ϕ_1} on this B_3 is free and gives a solid ring $S^1 \times B_3$, and this is the topology of the worldvolume. The 'transverse' sphere S^{n+1} is fibered over this worldvolume, shrinking to zero size at the ball boundaries, so the full horizon is topologically $S^1 \times S^{n+4}$, *i.e.,* a black ring, but one in which three of the directions of the S^{n+4}, namely those along r_1 and (r_2, ϕ_2), are much longer than the others and can be comparable to the S^1 length along ϕ_1. We call this again a prolate black ring. Note the following limits: when the inequality (6.22) is saturated the lens degenerates to the point ($r_1 = 1/\Omega_1$, $r_2 = 0$), and we recover a conventional dipole black ring; when $\Phi_H = 0$ the solid ring becomes a ball B_4 and we recover a neutral ultraspinning MP black hole; when $\Omega_2 \to 0$ we recover the annulus of the previous subsection times the plane (r_2, ϕ_2); when $\Omega_1 \to 0$ the strings and the rotation lie on orthogonal planes: the rotation limits the maximum size of the blackfold to $r_2 \leq 1/\Omega_2$, while the strings put a minimum to $r_1 = \Phi_H/(2\pi\sqrt{N})$ on the plane $r_2 = 0$, but r_1 is unbounded above so this does not describe a compact black hole.

By varying the parameters of the solutions, one can deform the hollow ball into the solid ring solution. The calculation of the charges of these solutions is straightforward and we will not undertake it, nor will we embark in the study of the multiple shape these blackfolds can assume for $p > 4$ with generic parameters. We simply remark that the construction in this section generalizes to blackfolds wrapping the direct product of any odd-sphere and any odd-ball, as it can be seen by taking equal rotation on all $k = p/2$ planes, $\Omega_1 = \ldots = \Omega_k = \Omega$, and the polarization lying on the first m planes, $\varsigma_1 = \ldots = \varsigma_m = 1$ and $\varsigma_{m+1} = \ldots = \varsigma_k = 0$. Defining $\rho_1^2 = r_1^2 + \cdots + r_m^2$ and $\rho_2^2 = r_{m+1}^2 + \cdots + r_k^2$, the positive quadrant in the (ρ_1, ρ_2) plane represents the quotient $\mathbb{R}^p/(SO(2m) \times SO(p-2m))$ and the fluid is again confined in a half-lens-shaped region of this plane. Then, reasoning along the same lines as above, it follows that the blackfold has topology $S^{2m-1} \times B_{p-2m+1}$, and the corresponding black hole has a $S^{2m-1} \times S^{n+p+2-2m}$ event horizon. The first factor is an odd-sphere, but the solution differs from those we analyze next in that the second spherical factor is prolate, much longer along $p - 2m + 1$ directions than along the others.

6.3 Odd-sphere solution with string dipole

The blackfold equations can easily be solved for blackfolds wrapping a round odd-sphere S^p, $p = 2k+1$, of radius R in a background Minkowski spacetime[10]. The construction is very similar to that in sec. 5.3, with the same choice for the velocity field u. The string polarization vector is

$$v = \frac{\gamma}{R}\left(\sum_{i=1}^{k+1}\frac{\partial}{\partial\phi_i} + \Omega R^2\frac{\partial}{\partial t}\right) \tag{6.25}$$

and the worldsheet area element

$$|\hat{h}|^{1/2} = R\,. \tag{6.26}$$

The potential is constant over the sphere

$$\Phi = \frac{\Phi_H}{2\pi R}\,, \tag{6.27}$$

and so are r_0 and α, too,

$$r_0 = \frac{n}{4\pi T}\sqrt{1-\Omega^2R^2}\,\left(1-\frac{\Phi_H^2}{N(2\pi R)^2}\right)^{N/2}\,, \tag{6.28}$$

$$\tanh\alpha = \frac{\Phi_H}{\sqrt{N}2\pi R}\,. \tag{6.29}$$

The extrinsic curvatures $K^r = -p/R$ and $\hat{K}^r = -1/R$, give easily the solution to the extrinsic equations (2.42)

$$R = \frac{1}{\Omega}\sqrt{\frac{p+nN\sinh^2\alpha}{n+p+nN\sinh^2\alpha}}\,. \tag{6.30}$$

The product ΩR is an increasing function of the charge (and the potential), ranging from the neutral limit $\Omega R = \sqrt{p/(n+p)}$ to lightlike rotation $\Omega R = 1$ in the extremal limit $\alpha \to \infty$, $\Phi_H \to \sqrt{N}$. This behavior is as expected from our discussion at the end of sec. 2.3: the addition of strings increases the tension so in a sphere of fixed radius R the angular velocity must be larger the more strings there are. The expression (6.30) takes a simpler form in terms of the rapidity η such that $\tanh\eta = \Omega R$,

$$n\sinh^2\eta = p + nN\sinh^2\alpha\,. \tag{6.31}$$

Note however that in order to obtain the radius as function of Ω and Φ_H we must substitute (6.29) in (6.30) and solve, to find

$$R = \frac{\sqrt{4\pi^2pN + (n+p-nN)\Omega^2\Phi_H^2 + \sqrt{\left(4\pi^2pN-(n+p-nN)\Omega^2\Phi_H^2\right)^2 + 4n^2N^2\Omega^2\Phi_H^2}}}{2\pi\Omega\sqrt{2N(n+p)}}\,. \tag{6.32}$$

[10] When $p=1$ these solutions are a particular case of those considered in [17].

This result can also be obtained from the variation of the action.

We can find the thickness in terms of T, Ω and the potential Φ_H, or more conveniently of α, as

$$r_0 = \frac{n^{3/2}}{4\pi T}\left(n + p + nN\sinh^2\alpha\right)^{-1/2}(\cosh\alpha)^{-N} \,. \tag{6.33}$$

Since the worldvolume fields are constant, it is straightforward to perform the integrals and obtain the thermodynamic variables for these blackfolds: M, J, S involve simple multiplication by the size $R^p\Omega_{(p)}$ of the sphere S^p, while for Q we have to multiply by the size of the subspace orthogonal to the strings. Since these are aligned with the diagonal of the Cartan subgroup of S^p, $p = 2k+1$, this subspace is a $\mathbb{C}\mathrm{P}^k$ of radius R with size $R^{p-1}\Omega_{(p)}/(2\pi)$. Then

$$M = \frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{1}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{\left(p + nN\sinh^2\alpha\right)^{p/2}\left(n + p + 1 + 2nN\sinh^2\alpha\right)}{\left(n + p + nN\sinh^2\alpha\right)^{(n+p)/2}(\cosh\alpha)^{nN}}, \tag{6.34}$$

$$J = \frac{\Omega_{(n+1)}\Omega_{(p)}}{16\pi G}\frac{1}{\Omega^{p+1}}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{\left(p + nN\sinh^2\alpha\right)^{(p+2)/2}}{\left(n + p + nN\sinh^2\alpha\right)^{\frac{n+p}{2}}(\cosh\alpha)^{nN}}, \tag{6.35}$$

$$S = \frac{\Omega_{(n+1)}\Omega_{(p)}}{4G}\frac{1}{\Omega^p}\left(\frac{n^{3/2}}{4\pi T}\right)^{n+1} \frac{\left(p + nN\sinh^2\alpha\right)^{p/2}}{n^{1/2}\left(n + p + nN\sinh^2\alpha\right)^{(n+p)/2}(\cosh\alpha)^{nN}}, \tag{6.36}$$

$$Q = \frac{\Omega_{(n+1)}\Omega_{(p)}}{32\pi^2 G}\frac{1}{\Omega^{p-1}}\left(\frac{n^{3/2}}{4\pi T}\right)^n \frac{n\sqrt{N}\sinh\alpha\left(p + nN\sinh^2\alpha\right)^{(p-1)/2}}{\left(n + p + nN\sinh^2\alpha\right)^{(n+p-1)/2}(\cosh\alpha)^{nN-1}}, \tag{6.37}$$

with α given by the potential Φ_H in (6.29). Here J is the angular momentum associated to $\sum_{i=1}^{k+1}\partial_{\phi_i}$.

Again, we find that the entropy of these black holes scales with the mass and angular momentum according to (3.57). As for the annulus solution, the ratio Q/M is proportional to Ω times a bounded function of Φ_H only, and therefore there is no bound on the ratio of dipole Q to mass M.

The horizon of these solutions is $S^{2k+1}\times S^{n+1}$, and thus topologically the same as in the previous subsection. However, here the last sphere factor is of much smaller size than the first one in all directions. This small S^{n+1} is geometrically round to leading order in the blackfold approximation, but will be slightly distorted at the next order. As before, the large S^{2k+1} is exactly round as long as all the rotations Ω_i are equal, but this can be relaxed.

When $n = 1$, $p = 1$, these solutions can be compared to the exact five-dimensional dipole rings of [13] in the limit of large angular momentum. In appendix C.3 we show that there is perfect agreement.

Extremal limit. This is obtained as $\alpha \to \infty$, *i.e.*, $\Phi_H \to \sqrt{N}R$, so the rotation becomes lightlike,

$$R\Omega \to 1\,. \tag{6.38}$$

In order to keep the mass, spin and dipole charge finite we must also keep

$$\frac{(\sinh\alpha)^{2-n(N+1)}}{T^n} \quad \text{finite}\,, \tag{6.39}$$

which means that if $n(N+1) > 2$ then the temperature goes to zero. Since $n \geq 1$, this condition will not be met only if $N < 1$, which does not seem to be relevant in string theory. The marginal case where the temperature remains finite is $n = 1$, $N = 1$, which includes the well-known case of five-dimensional string-like objects with $N = 1$ and their direct uplifts. The condition depends only on the number of dimensions transverse to the brane (given by n) and of N, and is independent of p.

In this limit we recover the relations (4.11) for M, J and Q. The entropy of the extremal black holes is finite and non-vanishing only if $n(N-1) = 2$, which can happen only when $(n = 1, N = 3)$, or $(n = 2, N = 2)$. These cases are familiar when $p = 1$, where they corresponds to extremal black strings and rings in five and six dimensions, respectively. Other values of p are direct uplifts of them. In all these cases the entropy is equal to

$$S = \frac{4\pi}{\sqrt{n(n+2)}} \left(\frac{(n+2)\Omega_{(n+1)}\Omega_{(p)}}{16\pi G} \right)^{-1/n} R^{-\frac{p}{n}} \left(\frac{M}{2} \right)^{\frac{n+1}{n}} \tag{6.40}$$

As we discussed in the previous subsection, this entropy does not match that of the extremal annulus solutions. Nevertheless one can see that the curves of $S/M^{(n+1)/n}$ as function of Φ tend to cross around the region where the annulus entropy starts to diverge.

Finally, it is straightforward to construct solutions with string dipoles where the worldvolume is a product of odd-spheres. As they do not add anything qualitatively new, we omit the details.

7 Discussion and outlook

We begin by discussing separately our results for each of the two types of q-brane charge that we have considered in this paper, highlighting some particular consequences and conjectures motivated by our study.

Charged rotating black holes. We have constructed them in theories with arbitrary dilaton coupling in any dimension $D \geq 5$. In $D = 5$ we have found them as black rings, and in $D \geq 6$ we have found them with spherical topology, but also with more varied topologies, in particular products of spheres. After the results of [19, 21] it may not be so surprising to find this type of black holes when their rotation is very large: a sufficiently large rotation can conceivably support the tension of an extended brane with a compact worldvolume, and the addition of charge should not change this qualitatively. Nevertheless, it is illustrative of the power of the method that it has easily provided a window into the elusive charged rotating black holes of Einstein-Maxwell(-dilaton) theory with spherical topology, extending the Myers-Perry rotating black holes to have charge. Their five-dimensional counterparts, although expected to exist for any value of the dilaton, cannot become brane-like and therefore fall outside the applicability of the blackfold techniques. In five dimensions only string-like rings can be described as blackfolds.

A bigger surprise has been to find another regime, not involving large spins, in which the tension of the brane can be balanced: when the charge is near extremality, the forces on the brane tend to

cancel leaving only a very small tension, and therefore just a small rotation is needed to oppose it. We have found that this can happen when the rotation occurs in a number s, $1 \leq s < (D-3)/2$, of all independent rotation planes. This is a regime in which the black hole can become brane-like and therefore is amenable to study as a blackfold.

There is a remarkable conclusion of this result. The static extremal solutions, being charged dusts, are marginally stable solutions. However, the addition of a small non-extremality, which gives them a non-singular horizon, will make them unstable. This is because these blackfolds are locally black branes with 0-brane charge, and these are known to suffer Gregory-Laflamme instabilities whenever they are non-extremal (more on this below). In particular, this implies that in the Einstein-Maxwell-dilaton theories (1.1) with $q = 0$ and arbitrary a,

- Electrically charged black holes rotating in s planes, with $1 \leq s < (D-3)/2$, become dynamically unstable when their charge is sufficiently close to (but below) the extremal bound $Q = M/\sqrt{N}$. This applies in particular to black holes with horizons of spherical topology.

This may come as a surprise, since one would regard these topologically-spherical rotating charged black holes as the natural higher-dimensional counterparts of the Kerr-Newman solution, which is expected to be stable, in particular at slow rotation. Moreover, closeness to a BPS state might have suggested improved stability. What changes these expectations in $D \geq 6$ is that rotation in $s \in [1, (D-3)/2)$ planes close to extremality makes the black hole spread along these planes and approach the geometry of an unstable near-extremal black brane. This kind of instability will also be present for charged black holes with any other horizon topology, but since these are less familiar, their instability may seem less unexpected. The case where $s = (D-3)/2$ is not included in the statement above, but in that case charged black holes with product-of-spheres topology (secs. 5.3, 5.4), and in fact any that are amenable to the blackfold approach, are expected to be unstable close to extremality since locally they resemble unstable thin black branes. These include in particular the five-dimensional charged black rings with a single angular momentum[11]. The five-dimensional topologically-spherical charged black holes with one spin do not become brane-like and cannot be constructed as blackfolds, but if they get highly distorted near extremality they could plausibly be unstable to R-mode deformations like in the neutral case [32, 35].

Let us emphasize that this result is a conclusion, not a conjecture, that follows from the perturbative blackfold construction of these black holes. It allows us to establish that the instabilities will be eventually present when the charge is sufficiently large, but not to determine the precise parameter values for their onset. Following [32], we expect this to occur before the near-extremal brane-like regime is reached, and to be approximately marked by the criterion that it is entropically favorable to split the black hole, in a range of parameters after the appearance of a negative mode of the thermodynamic Hessian [36, 37]. One expects also the appearance of a new phase of black holes with pinches along the horizon. It should be interesting to study this phenomenon in more detail, in particular in the solutions with KK charge which are known in closed form.

[11] As discussed above, the presence of independent dipoles, or a Chern-Simons term, can take us away from the regime where this conclusion applies.

String-dipole black holes. An important motivation that we had to study these configurations was to try to find a black hole with a horizon of spherical topology supporting a string dipole. It is known for many of the theories in (1.1) that a static black hole cannot possess such dipole hair [15]. One might have hoped that a sufficiently large rotation could change this by spreading the horizon along the plane of rotation, and therefore we have attempted to construct the solution that could possibly give it, namely a disk blackfold. However, such a disk could be regarded as a concentric distribution of dipole rings, and if these must rotate rigidly, then at a small enough radius the centrifugal force will become too small to balance the tension of the strings. This forces the disk to open up a hole, so it becomes an annulus. The black hole turns then into a black ring.

This suggests that a horizon with spherical topology, no matter how large its rotation, cannot support string dipole hair, nor, quite likely, any brane dipole hair. We are therefore led to conjecture that

- Black hole horizons of spherical topology cannot support brane-dipole hair.

A still plausible but stronger conjecture is that

- Black hole horizons must have a non-contractible q-cycle in order to be able to support q-brane dipole hair.

These conjectures refer to asymptotically flat black holes of the theories (1.1). Other fields, or a cosmological constant, might significantly alter the physics of the system.

Like in the case of black holes built as neutral blackfolds, most of the solutions in this paper are unstable to perturbations that create ripples along the worldvolume [38]. This Gregory-Laflamme-type instability of objects that are locally like black p-branes can be suppressed near extremality along directions parallel to a q-brane current, but will persist in directions transverse to the current [39]. Thus the only configurations in this paper that are expected to be stable close to extremality are dipole rings built as circular strings. The prolate dipole black rings will be unstable, even close to extremality, to rippling along the elongated directions. In general the instabilities should mark the bifurcation into new phases with pinched horizons.

In this article we have only sampled solutions that illustrate the new possibilities. Generalizations of other solutions in [21] to include electric charges and dipoles should be straightforward. In particular, it is easy to find for both cases helical black rings that have the minimal horizon symmetry allowed by rigidity theorems.

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A p-brane solutions with diluted electric q-brane charges

In this appendix we build the generic black p-brane solution, carrying q-brane charge and with horizon topology $S^{n+1} \times \mathbb{R}^p$. We construct them starting from the charged, spherically symmetric black holes of $d = n + 3$ dimensional Einstein-Maxwell dilaton (EMD) theory, proceeding in two steps. First, we uplift these solution to $n + q + 3$ dimensions, to obtain q-branes charged under a $q + 1$-form gauge potential isotropic in the q extended dimensions. In a second uplift, we add $p - q$ extra extended dimensions, to obtain the desired solution.

More specifically, we start from EMD theory in d dimensions with dilaton coupling $\tilde{a}$ and action

$$I = \frac{1}{16\pi G} \int d^d x \sqrt{-\tilde{g}} \left[\tilde{R} - 2 \left(\tilde{\nabla} \tilde{\phi} \right)^2 - \frac{e^{-2\tilde{a}\tilde{\phi}}}{4} \tilde{F}^2 \right] . \tag{A.1}$$

Here, we are using the tilde to mark the d-dimensional quantities and contrast them to the corresponding quantities of the uplifted solution. The Maxwell field strength is $\tilde{F} = \mathrm{d}\tilde{B}_{[1]}$, with $\tilde{B}_{[1]}$ being the electromagnetic potential one-form.

The spherically symmetric, electrically charged black hole solutions of this theory where obtained by Gibbons and Maeda in [40]. Their metric can be put in the form

$$d\tilde{s}^2 = -\frac{f}{h^{\tilde{A}}} dt^2 + h^{\tilde{B}} \left(\frac{dr^2}{f} + r^2 d\Omega_{n+1}^2 \right) \tag{A.2}$$

with the functions $f(r)$, $g(r)$ and the coefficients $\tilde{A}$, $\tilde{B}$ given by

$$f(r) = 1 - \frac{r_0^n}{r^n} , \quad h(r) = 1 + \frac{r_0^n}{r^n} \sinh^2 \alpha , \quad \tilde{A} = \frac{4n}{2n + (n+1)\tilde{a}^2} , \quad \tilde{B} = \frac{4}{2n + (n+1)\tilde{a}^2} . \tag{A.3}$$

This metric describes a spherical black hole, with an event horizon located at $r = r_0$ and an electric charge determined by the parameter α. The corresponding dilaton field and Maxwell potential read

$$\tilde{\phi}(r) = -\frac{(n+1)\tilde{a}}{2n + (n+1)\tilde{a}^2} \ln h(r) , \qquad \tilde{B}_{[1]} = -\tilde{\Phi}(r) \mathrm{d}t , \tag{A.4}$$

where we have defined the electric potential

$$\tilde{\Phi}(r) = 2 \sqrt{\frac{n+1}{2n + (n+1)\tilde{a}^2}} \frac{r_0^n}{r^n h(r)} \sinh \alpha \cosh \alpha. \tag{A.5}$$

The mass and electric charge of this black hole are

$$M = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + Nn \sinh^2 \alpha \right) , \qquad Q = \frac{\Omega_{(n+1)}}{16\pi G} n \sqrt{N} r_0^n \sinh \alpha \cosh \alpha, \tag{A.6}$$

and their ratio is bounded from above according to

$$\frac{Q}{M} \leq \frac{1}{\sqrt{N}}. \tag{A.7}$$

Extremal black holes saturate this bound.

A.1 Electrically charged q-branes

Let us consider General Relativity in $D = d + q$ dimensions, coupled to a dilaton field ϕ and a $(q+1)$-form gauge potential $B_{[q+1]}$, with associated $(q+2)$-form field strength $H_{[q+2]} = \mathrm{d}B_{[q+1]}$. The corresponding action is (1.1). We want to construct p-brane solutions of this theory by uplifting the Gibbons-Maeda solution. To this end, we start with a metric formed by the warped product of a d-dimensional base space with metric $\tilde{g}_{\mu\nu}$ and coordinates x^μ, and q flat dimensions with coordinates y^m,

$$ds^2 = e^{2\alpha\phi(x)}\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\beta\phi(x)}\delta_{mn}(y)dy^m dy^n, \tag{A.8}$$

and an electric ansatz for the gauge potential with the q extra indices lying in the extended directions,

$$B_{[q+1]} = \tilde{B}_{[1]} \wedge \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^p, \tag{A.9}$$

in such a way that, upon dimensional reduction, the $(q + 1)$-form potential manifests itself as a Maxwell potential. We also take the dilaton field to be proportional to its lower dimensional embodiment, $\phi = \gamma\tilde{\phi}$, with γ a coefficient to be determined. The D-dimensional Ricci scalar R of the metric (A.8) can be written in terms of the Ricci scalar $\widetilde{R}$ of the base space and the dilaton ϕ as,

$$\begin{aligned} e^{2\alpha\phi}R = \widetilde{R} &- 2\left((d-1)\alpha + q\beta\right)\Delta\phi \\ &- \left((d-1)(d-2)\alpha^2 + 2q(d-2)\alpha\beta + q(q+1)\beta^2\right)(\nabla\phi)^2, \end{aligned} \tag{A.10}$$

Then, after imposing

$$(n+1)\alpha + q\beta = 0 \tag{A.11}$$

to enforce the Einstein frame, using the relation

$$H^2 = \frac{1}{2}(q+2)!\, e^{-4\alpha\phi-2q\beta\phi}\tilde{F}^2, \tag{A.12}$$

and integrating by parts a laplacian of the dilaton, the action (1.1) reduces to

$$I = \frac{1}{16\pi G}\int d^dx\sqrt{-\tilde{g}}\left[\tilde{R} - \left(\frac{n+q+1}{n+1}q\beta + 2\right)\gamma^2\left(\partial\tilde{\phi}\right)^2 - \frac{e^{-2(a+\alpha+q\beta)\gamma\tilde{\phi}}}{4}\tilde{F}^2\right], \tag{A.13}$$

with $\tilde{F} = \mathrm{d}\tilde{B}_{[1]}$. This action is the EMD action (A.1) with dilaton coupling $\tilde{a}$ given by

$$\tilde{a} = \gamma\left(a + \frac{nq}{n+1}\beta\right), \tag{A.14}$$

as long as one chooses γ such that the dilaton kinetic term has the correct normalization,

$$\left(\frac{n+q+1}{n+1}q\beta^2+2\right)\gamma^2=2. \tag{A.15}$$

However this is not sufficient to ensure that solutions of the lower dimensional EMD equations solve also the D-dimensional equations of motion. Indeed, varying equation (1.1) we find

$$R_{\mu\nu}=2\nabla_\mu\phi\nabla_\nu\phi+\frac{1}{2(q+1)!}e^{-2a\phi}H_{\mu\rho_1\ldots\rho_{q+1}}H_\nu{}^{\rho_1\ldots\rho_{q+1}}-\frac{q+1}{2(D-2)(q+2)!}e^{-2a\phi}H^2g_{\mu\nu}\,, \tag{A.16}$$

$$\Delta\phi+\frac{a}{4(q+2)!}e^{-2a\phi}H^2=0,\qquad \nabla_\nu\left(e^{-2a\phi}H^{\nu\mu_1\ldots\mu_{q+1}}\right)=0. \tag{A.17}$$

While the field equation for H is automatically satisfied for the uplifted solution, Einstein's equations and the dilaton equantion are not; indeed the equation for the dilaton becomes upon dimensional reduction

$$\tilde{\Delta}\tilde{\phi}+\frac{a}{8\gamma}e^{-2\tilde{a}\tilde{\phi}}\tilde{F}^2=0, \tag{A.18}$$

which is equivalent to the lower dimensional dilaton equation of motion if and only if $a=\gamma\tilde{a}$. This relation, with (A.14), can be solved to fix the value of β such that both the dilaton equation and the Einstein's equation are verified,

$$\beta=\frac{2n}{(n+q+1)a}\,. \tag{A.19}$$

With the latter relation, equations (A.11), (A.14) and (A.15) can be solved for α, γ and $\tilde{a}$. In particular, the dilaton couplings are related by

$$\tilde{a}^2=a^2+\frac{2n^2q}{(n+1)(n+q+1)}\,. \tag{A.20}$$

We can now uplift the Gibbons-Maeda solution to the higher dimensional theory, to generate electrically charged q-branes in $n+q+3$ dimensions. The resulting metric is

$$ds^2=-\frac{1}{h^A}\left(f\,dt^2+d\vec{y}^2\right)+h^B\left(\frac{dr^2}{f}+r^2d\Omega^2_{(n+1)}\right), \tag{A.21}$$

with

$$A=\frac{4n}{2n(q+1)+(n+q+1)a^2}\,,\qquad B=\frac{4(q+1)}{2n(q+1)+(n+q+1)a^2}\,, \tag{A.22}$$

the gauge field reads

$$B_{[q+1]}=-\sqrt{A+B}\,\frac{r_0^n}{r^nh(r)}\sinh\alpha\cosh\alpha\,\mathrm{d}t\wedge\epsilon\,, \tag{A.23}$$

and the dilaton is given by

$$\phi=-\frac{1}{4}\left(A+B\right)a\ln h(r)\,. \tag{A.24}$$

For future reference, notice that the A and B coefficients enter these formulas through the combination $A+B$, that we dub hereafter N, and that they satisfy the relation $(q+1)A-nB=0$.

A.2 Black p-branes with electric q-brane charge

We can now obtain general p-brane solutions with diluted q-brane charge ($q \leq p$) by taking the solution of the previous subsection, and uplifting it once more by adding $p-q$ additional extended directions. We keep the gauge field lying in the first q extended directions $y^1, \ldots, y^q$, and we rename the new extra $(p-q)$ coordinates as z^a. We first perfom a dimensional reduction along the y^m directions. Since the procedure is precisely the same as the one explained in the previous subsection, we will give the final solution without any further detail.

The black p-brane solution with electric q-charge to the theory defined by the action (1.1) in $D = n+p+3$ dimensions has the metric

$$ds^2 = -\frac{1}{h^A}\left(f\,dt^2 + d\vec{y}^2\right) + h^B\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(n+1)} + d\vec{z}^2\right), \tag{A.25}$$

with

$$A = \frac{4(n+p-q)}{2(q+1)(n+p-q)+(n+p+1)a^2}, \qquad B = \frac{4(q+1)}{2(q+1)(n+p-q)+(n+p+1)a^2}. \tag{A.26}$$

Then, defining

$$N = A + B = \frac{4(n+p+1)}{2(q+1)(n+p-q)+(n+p+1)a^2}, \tag{A.27}$$

the gauge field reads

$$B_{[q+1]} = -\sqrt{N}\,\frac{r_0^n}{r^n h(r)}\sinh\alpha\cosh\alpha\,\mathrm{d}t\wedge\mathrm{d}y^1\wedge\ldots\wedge\mathrm{d}y^q, \tag{A.28}$$

and the dilaton field is given by

$$\phi = -\frac{1}{4}Na\,\ln h(r). \tag{A.29}$$

Note that this solution is also valid for non-dilatonic branes. In this case, one simply has to use the previous formulas with $a=0$, giving:

$$A = \frac{2}{q+1}, \qquad B = \frac{2}{n+p-q}, \qquad N = \frac{2(n+p+1)}{(q+1)(n+p-q)}. \tag{A.30}$$

It is worth mentioning a few other particular cases. When $p=q$, one correctly recovers the solution of the previous subsection. Instead, setting $q=0$ one obtains general p-brane solutions of EMD theory,

$$A = \frac{4(n+p)}{2(n+p)+(n+p+1)a^2}, \quad B = \frac{4}{2(n+p)+(n+p+1)a^2}, \quad N = \frac{4(n+p+1)}{2(n+p)+(n+p+1)a^2}, \tag{A.31}$$

that reduce by choosing vanishing dilaton coupling a^{EM} to the p-branes of pure Einstein-Maxwell theory,

$$A^{\mathsf{EM}} = 2, \qquad B^{\mathsf{EM}} = \frac{2}{n+p}, \qquad N^{\mathsf{EM}} = 2\frac{n+p+1}{n+p}, \tag{A.32}$$

while, by choosing for the dilaton coupling

$$a^{\mathsf{KK}} = \sqrt{\frac{2(n+p+2)}{n+p+1}} = \sqrt{\frac{2(D-1)}{(D-2)}} \tag{A.33}$$

relevant for the Kaluza-Klein theory, one obtains

$$A^{\mathsf{KK}} = \frac{n+p}{n+p+1}, \qquad B^{\mathsf{KK}} = \frac{1}{n+p+1}, \qquad N^{\mathsf{KK}} = 1\,. \tag{A.34}$$

Finally, the metric can be written in a manifestly Poincaré invariant form to describe boosted branes,

$$ds^2 = \frac{1}{h^A}\left(\eta_{ab} + \frac{r_0^n}{r^n}u_a u_b + \left(h^N - 1\right)\hat{\perp}_{ab}\right)d\sigma^a d\sigma^b + h^B\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(n+1)}\right), \tag{A.35}$$

$$B_{[q+1]} = -\sqrt{N}\,\frac{r_0^n}{r^n h(r)}\sinh\alpha\cosh\alpha\, u\wedge v^{(1)}\wedge\ldots\wedge v^{(q)}\,, \tag{A.36}$$

where $\sigma^a = (t, \vec{y}, \vec{z})$, η_{ab} is the flat Minkowski metric, $\hat{\perp}_{ab}$ is the projector to the directions orthogonal to the gauge field, and $v^{(1)}, \ldots, v^{(q)}$ are an orthonormal basis of the spatial subspace of Minkowski on which the gauge field lies, $\eta_{ab}v^{(i)a}v^{(j)b} = \delta^{ij}$, $\hat{\perp}_{ab}v^{(i)b} = 0$.

B Charges and thermodynamics of the black branes

In this section, we compute the charges and study the thermodynamics of the black p-branes with q-brane charges obtained in the previous section. This will provide the basic input for the blackfold formalism: the equation of state of the effective fluid describing black holes with more than one scale. The mass density and the tension of the brane are obtained from the Brown-York stress tensor at infinity, that we regularize through the background subtraction technique. The electric charge density is obtained from the flux at infinity using the Gauss' law, while the temperature and entropy are derived in the standard way, resulting in an equation of state and in the thermodynamic relations for an anisotropic fluid.

Since we are computing worldvolume densities, these are the same as conserved charges of the original Gibbons-Maeda black hole that these branes are an uplift of. Thus the results should depend only on the parameters n and N which are invariant under compactification, and not on p nor q. Our results do indeed verify this.

B.1 Brown-York stress tensor: energy density and tension

The Brown-York quasi-local stress tensor is defined on a boundary timelike hypersurface, which we take at a fixed large radius $r = R$, and reads

$$16\pi G\tau_{\mu\nu} = K_{\mu\nu} - h_{\mu\nu}K - \left(\hat{K}_{\mu\nu} - h_{\mu\nu}\hat{K}\right), \tag{B.1}$$

where $K_{\mu\nu}$ is the extrinsic curvature of the boundary and the hatted variables are the corresponding quantities that we subtract, computed on flat spacetime with the same intrinsic geometry on both boundaries. It is easy to see that for a metric of the form

$$ds^2 = -\frac{f}{h^A}\,dt^2 + h^B\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(n+1)}\right) + h^C d\vec{y}^2 + h^D d\vec{z}^2, \tag{B.2}$$

the Brown-York stress tensor reads

$$\begin{aligned}
\tau_{tt} &= \frac{1}{16\pi G}\frac{r_0^n}{R^{n+1}}\left(n+1+(qC+(p-q)D+(n+1)B)\,n\sinh^2\alpha\right) + \mathcal{O}\left(\frac{1}{R^{2n+1}}\right),\\
\tau_{mn} &= -\frac{r_0^n}{16\pi G R^{n+1}}\left(1+((q-1)C+(p-q)D+(n+1)B-A)\,n\sinh^2\alpha\right)\delta_{mn} + \mathcal{O}\left(\frac{1}{R^{2n+1}}\right)\delta_{mn},\\
\tau_{ab} &= -\frac{r_0^n}{16\pi G R^{n+1}}\left(1+(qC+(p-q-1)D+(n+1)B-A)\,n\sinh^2\alpha\right)\delta_{ab} + \mathcal{O}\left(\frac{1}{R^{2n+1}}\right)\delta_{ab},\\
\tau_{ij} &= -\frac{n r_0^n}{16\pi G R^{n-1}}\left(qC+(p-q)D+nB-A\right)\sinh^2\alpha\,\sigma_{ij} + \mathcal{O}\left(\frac{1}{R^{2n-1}}\right)\sigma_{ij},
\end{aligned}$$

where the indices m, $n = 1,\ldots,q$ are along the q directions y^m, and the indices a, $b = 1,\ldots,p-q$ run along the $(p-q)$ directions z^a. For completeness we have also included the stress along the angular directions i,j of S^{n+1} with $d\Omega^2_{(n+1)} = \sigma_{ij}d\theta^i d\theta^j$.

We obtain the corresponding blackfold stress tensor by integrating over the transverse sphere at large R, and taking the limit $R\to\infty$ [19],

$$\begin{aligned}
T_{tt} &= \frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(n+1+(qC+(p-q)D+(n+1)B)\,n\sinh^2\alpha\right),\\
T_{mn} &= -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(1+((q-1)C+(p-q)D+(n+1)B-A)\,n\sinh^2\alpha\right)\delta_{mn},\\
T_{ab} &= -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(1+(qC+(p-q-1)D+(n+1)B-A)\,n\sinh^2\alpha\right)\delta_{ab}.
\end{aligned} \tag{B.3}$$

Notice that the integrated tension of the S^{n+1} sphere diverges unless

$$qC+(p-q)D+nB-A = 0. \tag{B.4}$$

This is automatically satisfied by all the solutions obtained in appendix A, and we will assume it in the following.

This stress tensor describes an anisotropic perfect fluid in the $(p+1)$-dimensional worldvolume of the blackfold with energy density and pressures $P_\parallel$ in the directions of the gauge field and $P_\perp$ in the orthogonal directions to it along the extended dimensions given respectively by

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(n+1+nN\sinh^2\alpha\right), \tag{B.5}$$

$$P_\parallel = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(1+(B-C)\,n\sinh^2\alpha\right), \tag{B.6}$$

$$P_\perp = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n\left(1+(B-D)\,n\sinh^2\alpha\right). \tag{B.7}$$

All extensive charges are defined as densities per unit worldvolume of the blackfold as measured from infinity, *i.e.,* we are formally dividing by the factor $\int d^q y d^{p-q} z$. In particular, for all solutions obtained in the previous section, $D = B$ and $C = -A$, and the pressures read

$$P_{\parallel} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1 + nN \sinh^2 \alpha\right), \qquad P_{\perp} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n. \tag{B.8}$$

As expected, in its perpendicular directions the gauge field does not produce any stress, and $P_{\perp}$ is the tension in the neutral case, while $P_{\parallel}$ gets modified by the gauge field stresses. They lead to the equation of state

$$\varepsilon = -P_{\parallel} - nP_{\perp}\,. \tag{B.9}$$

B.2 Electric q-brane charge and potential

As usual, we define the electric charge by measuring the electric flux at infinity. To fix the normalization, we will conventionally assume that the electromagnetic current $J^{\mu_1 \ldots \mu_{q+1}}$ source due to external matter couples to the gauge field $B_{[q+1]}$ according to the action

$$I_{sourced} = I + \frac{1}{(q+1)!} \int d^D x \sqrt{-g}\, B_{\mu_1 \ldots \mu_{q+1}} J^{\mu_1 \ldots \mu_{q+1}}. \tag{B.10}$$

where I is (1.1). Then, the equations of motion for the gauge field get modified to

$$\nabla_\nu \left[e^{-2a\phi} H^{\nu \mu_1 \ldots \mu_{q+1}} \right] = -8\pi G J^{\mu_1 \ldots \mu_{q+1}}. \tag{B.11}$$

Using Gauss' law it is then possible to define a conserved electric charge as a measure of the electric flux at infinity,

$$\mathcal{Q} = \oint * J = \frac{1}{8\pi G (q+2)!} \oint e^{-2a\phi} H^{\mu_1 \ldots \mu_{q+2}} dS_{\mu_1 \ldots \mu_{q+2}}, \tag{B.12}$$

with the integration measure given by

$$dS_{\mu_1 \ldots \mu_{q+2}} = (q+2)!\, v^{(1)}_{[\mu_1} \cdots v^{(q+2)}_{\mu_{k+1}]}\, dV_{(D-q-2)}, \tag{B.13}$$

where $dV_{(D-q-2)}$ is the volume element of the hypersurface through which the flux is measured, and the vectors are the unit normal vectors to this hypersurface. To evaluate it for the charged branes (A.25)-(A.29), we integrate over the $\mathcal{S}^{n+1} \times \mathbb{R}^{p-q}$ hypersuface defined at constant $(t, r, y^1, \ldots, y^q)$ and take the $r \to \infty$ limit. The integration over $\mathcal{S}^{n+1}$ yields a $\Omega_{(n+1)}$ factor, while we omit the formally infinite $\int dz^1 \cdots \int dz^{p-q}$ integral to obtain the charge density

$$\mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \sinh\alpha \cosh\alpha\,. \tag{B.14}$$

It is important here to note that this charge is a density *in the orthogonal directions to the gauge field*, *i.e.* to obtain the total charge one should not integrate over all p spacelike directions of the brane worldvolume, but only on the $p-q$ directions projected by $\hat{\perp}_{ab}$ in (A.35). Finally, the electric potential at the horizon, measured with respect to infinity, is defined as

$$\Phi = B_{ty^1 \ldots y^q}(\infty) - B_{ty^1 \ldots y^q}(r_0) = \sqrt{N} \tanh\alpha. \tag{B.15}$$

B.3 Anisotropic fluid thermodynamics

The temperature of these black objects is

$$T = \frac{n}{4\pi r_0} (\cosh\alpha)^{-N}\,, \tag{B.16}$$

and the entropy density is obtained from the area density,

$$s = \frac{a_H}{4G} = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} (\cosh\alpha)^N\,. \tag{B.17}$$

Notice that the temperature, entropy density, energy density (B.5), charge density (2.37) and the electric potential (2.36) depend only on the parameters n and N and not on p.

With these expressions for the energy, temperature, entropy density, charge density and electric potential, it is easy to check that the Smarr relation

$$\epsilon = \frac{n+1}{n} Ts + \Phi\mathcal{Q} \tag{B.18}$$

holds, as well as the first law of thermodynamics

$$\mathrm{d}\epsilon = T\mathrm{d}s + \Phi\mathrm{d}\mathcal{Q}\,. \tag{B.19}$$

It is worth mentioning that it is possible to obtain a simple expression for the Gibbs free energy $G(T,\Phi) = \epsilon - Ts - \Phi\mathcal{Q}$ in term of its natural variables,

$$G(T,\Phi) = \frac{\Omega_{(n+1)}}{16\pi G} \left(\frac{n}{4\pi T}\right)^n \left[1 - \frac{\Phi^2}{N}\right]^{nN/2}\,, \tag{B.20}$$

from which one can easily verify the relations

$$s = -\left.\frac{\partial G}{\partial T}\right|_{\Phi_H}\,, \qquad \mathcal{Q} = -\left.\frac{\partial G}{\partial \Phi}\right|_T\,. \tag{B.21}$$

For an anisotropic fluid of the kind we discuss we have the Gibbs-Duhem relations[12]

$$\epsilon + P_{\parallel} = Ts\,, \qquad \epsilon + P_{\perp} = Ts + \Phi\mathcal{Q}\,. \tag{B.22}$$

Together with the first law as written above, and the equation of state of the fluid, these equations determine the thermodynamics of the system. The Smarr relation above is a consequence of them.

From the first law and the Gibbs-Duhem relations we obtain

$$dP_{\perp} = sdT + \mathcal{Q}d\Phi\,, \qquad dP_{\parallel} = sdT - \Phi d\mathcal{Q}\,. \tag{B.23}$$

[12]Note that $P_{\perp}$ coincides with $-G$, while $-P_{\parallel} = \epsilon - Ts$ is the Helmoltz free energy of the system.

C Comparison with exact black hole solutions

C.1 Charged rotating MP black holes in KK theory

The general charged rotating solution of the Einstein-Maxwell dilaton theory (A.1) with dilaton coupling (A.33) was generated in [9] starting with the D-dimensional MP black hole with general angular momenta, uplifting it to pure Einstein-Maxwell theory in $D+1$ dimensions, boosting it along the extra dimension and reducing the resulting solution back to D dimensions, where the boost of the solution becomes an electric charge. Having an exact charged rotating explicit solution for this case will provide us with a setting in which we can test the blackfold approach by comparing the effective theory results with the exact results. The final solution has metric, gauge field and dilaton given by

$$ds^2 = \left(1 + \frac{mr^{2-\epsilon}}{\Pi F} \sinh^2 \alpha\right)^{\frac{1}{D-2}} \left[-dt^2 + \frac{\Pi F dr^2}{\Pi - mr^{2-\epsilon}} + \left(r^2 + a_i^2\right)\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) + \epsilon r^2 d\nu^2 \right.$$
$$\left. + \frac{mr^{2-\epsilon}}{\Pi F + mr^{2-\epsilon}\sinh^2 \alpha}\left(\cosh\alpha dt - a_i \mu_i^2 d\phi_i\right)^2\right], \qquad \text{(C.1)}$$

where repeated $i = 1 \ldots N$ indices are summed over, $N = \lfloor \frac{D-1}{2} \rfloor$ is the number of independent angular momenta[13] and $\epsilon = D - 2N - 1$ vanishes for odd D and $\epsilon = 1$ for even D. The direction cosines μ_i verify the constraint

$$\sum_{i=1}^{N} \mu_i^2 + \epsilon \nu^2 = 1. \qquad \text{(C.2)}$$

so that the coordinates (μ_i, ϕ_i) describe the N independent planes of the $D-2$ transverse sphere. The functions F and Π read respectively

$$F = 1 - \sum_{i=1}^{N} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \qquad \Pi = \prod_{i=1}^{N} (r^2 + a_i^2). \qquad \text{(C.3)}$$

Finally, the gauge field and the dilaton assume the form,

$$A = -\frac{mr^{2-\epsilon}\sinh\alpha}{\Pi F + mr^{2-\epsilon}\sinh^2\alpha}\left(\cosh\alpha\, \mathrm{d}t - a_i \mu_i^2\, \mathrm{d}\phi_i\right) \qquad \text{(C.4)}$$

$$\phi = -\frac{1}{4} a^{\mathsf{KK}} \ln\left(1 + \frac{mr^{2-\epsilon}}{\Pi F}\sinh^2\alpha\right) \qquad \text{(C.5)}$$

The ultraspinning limit for this solution can be obtained in the usual way, we send s rotation parameters a_j with $j = 1, \ldots, s$ to infinity, keeping the a_k with $k = s+1, \ldots, N$ finite, zooming into the region close to the poles by keeping the new coordinates $\sigma_j = a_j \mu_j$ finite in the limiting process, and rescaling the mass parameter such that $\hat{m}$, defined by

$$\frac{m}{\prod a_j^2} = \hat{m}, \qquad \text{(C.6)}$$

[13] We employ this fairly standard notation since there is no possibility of mistaking this N with (1.3): in this appendix we are concerned with Kaluza-Klein solutions for which N in (1.3) is 1.

remains finite. Then the (σ_j, ϕ_j) coordinates describe s (conformally) flat planes, and we use $p = 2s$ cartesian coordinates $\vec{y}$ to parameterize the corresponding $\mathbb{R}^p$. The resulting field configuration is

$$ds^2 = \left(1 + \frac{\hat{m}r^{2-\epsilon}}{\hat{\Pi}\hat{F}} \sinh^2 \alpha\right)^{\frac{1}{D-2}} \left[-dt^2 + \frac{\hat{\Pi}\hat{F}dr^2}{\hat{\Pi} - \hat{m}r^{2-\epsilon}} + \left(r^2 + a_k^2\right)\left(d\mu_k^2 + \mu_k^2 d\phi_k^2\right) + \epsilon r^2 d\nu^2 \right.$$
$$\left. + \frac{\hat{m}r^{2-\epsilon}}{\hat{\Pi}\hat{F} + \hat{m}r^{2-\epsilon}\sinh^2\alpha}\left(\cosh\alpha dt - a_k \mu_k^2 d\phi_k\right)^2 + d\vec{y}^2\right] \tag{C.7}$$

where k indices now range from $s + 1$ to N and the remaining direction cosines μ_k verify the constraint

$$\sum_{k=s+1}^{N} \mu_k^2 + \epsilon\nu^2 = 1. \tag{C.8}$$

so that the coordinates (μ_i, ϕ_i) describe the N independent planes of the $D - 2$ transverse sphere. The limiting functions $\hat{F}$ and $\hat{\Pi}$ read instead

$$\hat{F} = 1 - \sum_{k=s+1}^{N} \frac{a_k^2 \mu_k^2}{r^2 + a_k^2}, \qquad \hat{\Pi} = \prod_{k=s+1}^{N} (r^2 + a_k^2). \tag{C.9}$$

and the gauge and dilaton fields assume the form,

$$A = -\frac{\hat{m}r^{2-\epsilon}\sinh\alpha}{\hat{\Pi}\hat{F} + \hat{m}r^{2-\epsilon}\sinh^2\alpha}\left(\cosh\alpha\, \mathrm{d}t - a_k\mu_k^2\, \mathrm{d}\phi_k\right) \tag{C.10}$$

$$\phi = -\frac{1}{4} a^{\mathsf{KK}} \ln\left(1 + \frac{\hat{m}r^{2-\epsilon}}{\hat{\Pi}\hat{F}}\sinh^2\alpha\right). \tag{C.11}$$

These solutions represent black membranes, with p extended dimensions, whose horizon is $\mathbb{R}^p \times \mathcal{S}^{n+1}$, and with arbitrary angular momenta on the $(n+1)$-sphere (n is defined as $n = D - p - 3$).

As we shall not consider in this work the angular momentum on the sphere, we set $a_k = 0$ and we obtain, setting $\hat{m} = r_0^n$,

$$ds^2 = -\frac{f}{h^{\frac{n+p}{n+p+1}}}\, dt^2 + h^{\frac{1}{n+p+1}}\left(\frac{dr^2}{f} + d\vec{y}^2 + r^2 d\Omega_{n+1}^2\right) \tag{C.12}$$

with

$$A = -\frac{r_0^n \sinh 2\alpha}{2r^n h(r)}\, \mathrm{d}t, \qquad \phi = -\frac{1}{4} a^{\mathsf{KK}} \ln h(r). \tag{C.13}$$

The physical properties of the charged MP dilaton black hole are

$$M = \frac{\Omega_{(D-2)}}{16\pi G} m \left(1 + (D-3)\cosh^2\alpha\right), \qquad Q = \frac{(D-3)\Omega_{(D-2)}}{16\pi G} m \sinh\alpha\cosh\alpha, \tag{C.14}$$

$$J_i = \frac{\Omega_{(D-2)}}{8\pi G} m a_i \cosh\alpha, \qquad \Omega_i = \frac{a_i}{\left(r_+^2 + a_i^2\right)\cosh\alpha}, \tag{C.15}$$

$$\mathcal{A}_{\mathrm{h}} = \Omega_{(D-2)} \frac{\cosh\alpha}{r_+^{1-\epsilon}} \prod_{i=1}^{N}\left(r_+^2 + a_i^2\right), \qquad \kappa = \frac{1}{\cosh\alpha}\left(\sum_{i=1}^{N} \frac{r_+}{r_+^2 + a_i^2} - \frac{2-\epsilon}{2r_+}\right), \tag{C.16}$$

where r_+ is the largest non-negative root of $\prod_i (r^2 + a_i^2) - m r^{2-\epsilon} = 0$.

In the ultraspinning limit, with s ultraspins such that $a_i \gg m^{1/(D-3)}$, we obtain

$$M \to \frac{\Omega_{(D-2)}}{16\pi G} r_+^n \left(\prod_{i=1}^{s} a_i^2 \right) \left(1 + (D-3)\cosh^2\alpha \right) , \quad Q \to \frac{(D-3)\Omega_{(D-2)}}{16\pi G} r_+^n \left(\prod_{i=1}^{s} a_i^2 \right) \sinh\alpha \cosh\alpha \,, \tag{C.17}$$

$$J_i \to \frac{\Omega_{(D-2)}}{8\pi G} r_+^n a_i \cosh\alpha \prod_{j=1}^{s} a_j^2 \,, \qquad \Omega_i \to \frac{1}{a_i \cosh\alpha} \tag{C.18}$$

$$\mathcal{A}_{\mathrm{h}} \to \Omega_{(D-2)} r_+^{n+1} \cosh\alpha \prod_{i=1}^{s} a_i^2 \,, \qquad \kappa \to \frac{n}{2 r_+ \cosh\alpha} \,, \tag{C.19}$$

where we have defined

$$n = D - 2s - 3 \,. \tag{C.20}$$

Then, eliminating r_+ and a_i in favor of the temperature $T = \kappa/2\pi$ and Ω_i, we obtain in the ultraspinning regime,

$$\begin{aligned} M &\to \frac{\Omega_{(D-2)}}{16\pi G} \left(\frac{n}{4\pi T} \right)^n \frac{1 + (D-3)\cosh^2\alpha}{(\prod \Omega_i^2) \cosh^{D-3}\alpha}, & Q &\to \frac{(D-3)\Omega_{(D-2)}}{16\pi G} \left(\frac{n}{4\pi T} \right)^n \frac{\sinh\alpha\cosh\alpha}{(\prod \Omega_i^2)\cosh^{D-3}\alpha}, \\ J_i &\to \frac{\Omega_{(D-2)}}{8\pi G} \left(\frac{n}{4\pi T} \right)^n \frac{\cosh^{-(D-3)}\alpha}{\Omega_i \prod_j \Omega_j^2}, & \mathcal{A}_{\mathrm{h}} &\to \Omega_{(D-2)} \left(\frac{n}{4\pi T} \right)^{n+1} \frac{\cosh^{-(D-3)}\alpha}{\prod_i \Omega_i^2}. \end{aligned} \tag{C.21}$$

We can compare these results in the single spinning case with the blackfold results obtained in section 5.1. Taking $s = 1$, and using the relation

$$\Omega_{(D-2)} = \Omega_{(n+3)} = \frac{2\pi}{n+2} \Omega_{(n+1)} \tag{C.22}$$

the ultraspinning charges reduce to

$$\begin{aligned} M &= \frac{\Omega_{(n+1)}}{8G(n+2)\Omega^2} \left(\frac{n}{4\pi T} \right)^n \frac{1 + (n+2)\cosh^2\alpha}{(\cosh\alpha)^{n+2}}, & J &= \frac{\Omega_{(n+1)}}{4G(n+2)\Omega^3} \left(\frac{n}{4\pi T} \right)^n (\cosh\alpha)^{-(n+2)} \,, \\ \mathcal{A}_{\mathrm{h}} &= \frac{2\pi\Omega_{(n+1)}}{(n+2)\Omega^2} \left(\frac{n}{4\pi T} \right)^{n+1} (\cosh\alpha)^{-(n+2)} \,, & Q &= \frac{\Omega_{(n+1)}}{8G\Omega^2} \left(\frac{n}{4\pi T} \right)^n \frac{\sinh\alpha\cosh\alpha}{(\cosh\alpha)^{n+2}}. \end{aligned} \tag{C.23}$$

Substituting the parameter α for the electric potential defined by $\Phi_H = \tanh\alpha$ and the horizon area for $S = \mathcal{A}_{\mathrm{h}}/4G$, it can readily be checked these quantities, describing the ultraspinning regime of the MP dilaton black hole, coincide with the charges (5.11) obtained using the blackfold approach.

The ultimate meaning of the agreement between our blackfold construction of KK-charged ultraspinning black holes, and the ultraspinning limit of the exact rotating KK solutions, is that the procedure of adding KK charge by boosting in the KK direction commutes with the ultraspinning limit.

C.2 Charged black rings in five dimensions

The solution we consider is a particular case of those derived in [6, 7], and we use the notation of the latter reference. We set to zero two of the charge parameters α_i and all three independent dipole parameters μ_i. The solution will then have a charge and only the dipole that is induced by the rotation of the charge. Otherwise, the blackfold that would describe it should have both 0-brane and 1-brane charge, which is possible but is not addressed in this paper.

The solution is

$$\begin{aligned} ds^2 &= -h^{-2/3}\frac{F(y)}{F(x)}\left(dt + R(1+y)\frac{C_\lambda}{F(y)}\cosh\alpha d\psi\right)^2 \\ &\quad + h^{1/3}F(x)\frac{R^2}{(x-y)^2}\left(-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right), \end{aligned} \tag{C.24}$$

$$A = \frac{\lambda(y-x)}{hF(x)}\cosh\alpha\sinh\alpha\, dt + RC_\lambda\frac{1+y}{hF(x)}\sinh\alpha\, d\psi\,, \tag{C.25}$$

$$\phi = -\frac{1}{\sqrt{6}}\ln h\,, \tag{C.26}$$

where

$$F(\xi) = 1 + \lambda\xi\,, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \qquad h = 1 + \frac{\lambda(x-y)}{F(x)}\sinh^2\alpha\,, \tag{C.27}$$

$$C_\lambda = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,, \tag{C.28}$$

and the equilibrium of the ring requires

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \tag{C.29}$$

The ultraspinning, infinite radius limit of this solution is obtained by taking $R \to \infty$, $\lambda, \nu \to 0$ while keeping fixed α and

$$R/y = -r, \qquad \nu R = r_0\,, \qquad \lambda R = r_0(\cosh^2\eta + \sinh^2\eta\sinh^2\alpha) \tag{C.30}$$

(the apparent difference in the scaling of λ relative to [7] is due to the different order in which the momenta in the KK direction and in the string direction are taken, *i.e.,* the boosts are measured in different frames). In this limit the solution becomes the same as the boosted string with KK charge in (A.35) with $A = 2/3$, $B = 1/3$, $N = 1$. The equilibrium condition (C.29) now becomes

$$\cosh^2\eta + \sinh^2\eta\sinh^2\alpha = 2\,, \tag{C.31}$$

i.e.,

$$\sinh^2\eta\cosh^2\alpha = 1\,, \tag{C.32}$$

which is exactly the same as (5.22) with $p = n = N = 1$. Since in the limit $R \to \infty$ it has been shown that the mass, angular momentum, charge and entropy of the black ring are equal to the integrals of worldsheet densities [6], this agreement between the equilibrium boosts is sufficient to guarantee that the blackfold construction reproduces correctly all physical properties of the ultraspinning black ring.

We might also consider two non-zero charges but this does not add anything qualitatively new. The non-dilatonic solutions $N = 3$ in [7] appear to fall outside the scope of our construction due to the effect of the Chern-Simons term in the action.

C.3 Dipole black rings in five dimensions

Ref. [13] presented exact solutions for five-dimensional black rings with string dipole. It was also shown that in the ultraspinning regime the solution limits to a boosted black string with string charge, and with a specific value of the boost. This value is exactly the same that we have obtained in (6.31), for $n = 1$, $p = 1$, and any N. For the same reason as in the previous example of charged rings, it follows that our blackfold construction correctly reproduces the physics of five-dimensional dipole black rings to leading order in the ultraspinning regime.

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